

Instability of asymptotically anti de Sitter black holes under Robin conditions at the timelike boundary

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The static region outside the event horizon of an asymptotically anti de Sitter black hole has a conformal timelike boundary \mathcal{I} , the evolution from initial data of linear fields satisfying hyperbolic equations is a well posed problem only after imposing boundary conditions at \mathcal{I} . Boundary conditions preserving the action of the background isometry group on the solution space are limited to the homogeneous Dirichlet, Neumann or Robin types. We study, scalar and Maxwell fields and gravitational perturbations on asymptotically AdS black holes arising in Einstein and Lovelock theories. A decomposition in modes transforms the field equations into a set of wave equations with time independent potentials for auxiliary fields in the $x < 0$ half of 1+1 Minkowski spacetime. We study systematically these equations for the case of potentials not diverging at the boundary and prove that there is always an instability if Robin boundary conditions with large γ (the quotient between the derivative and the function at the boundary) are allowed. We show that Robin conditions on the auxiliary fields arise naturally both in classical gravity and in the AdS/CFT context. The mechanisms that trigger the Robin instabilities are explained, and it is shown that energy flux from \mathcal{I} is not a sufficient condition for instability. We also analyze the supersymmetric type of duality exchanging odd and even modes in four dimensional Schwarzschild anti de Sitter black holes and prove that this symmetry is broken except when a combination of Dirichlet conditions in the even sector and a particular Robin condition in the odd sector is enforced, or viceversa, although only the first of these two choices leads to a stable dynamics.

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I. INTRODUCTION

A preliminary stability criterion for a stationary black hole is that linear fields on the outer stationary region do not grow unbounded from initial data. Fields of interest are Klein Gordon, Maxwell and linear perturbations of the metric. The evolution of linearized metric perturbations is particularly important because it gives a hint about the ultimate question of full non linear stability, which is whether or not generic initial data (for the gravitational field equations) close to that of the black hole will evolve into spacetimes that asymptotically approach stationary black holes of similar characteristics. There are cases, however, where the complexity of the metric and the field equations make even an integral treatment of the linear gravity problem particularly complex (some examples are higher dimensional hairy black holes in generalized gravity theories.) In those cases the stability of scalar and/or Maxwell fields is often considered as indicative of linear gravity stability. In any case, the stability notion assumes unique evolution from initial data, which, given that the fields of interest obey hyperbolic equations, is guaranteed only if the outer region is globally hyperbolic. Asymptotically anti de Sitter spacetimes, however, are not globally hyperbolic, they have a conformal timelike boundary \mathcal{I} where boundary conditions have to be imposed to assure unambiguous evolution from initial data. When different choices of boundary conditions are possible (we will see below that this is not always the case), they lead to different dynamics outside the domain of dependence of the initial data hypersurface, and therefore to potentially different answers to the issue of stability.

Our primary example illustrating this situation is the four dimensional Schwarzschild anti de Sitter black hole solution (SAdS₄) of GR (GR), whose metric in static coordinates is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

where

$$f(r) = 1 - 2M/r - \Lambda r^2/3, \quad (2)$$

$\Lambda < 0$ is the cosmological constant and $M > 0$ the mass. Note that f has a unique positive root at $r = r_h$, the horizon radius, in terms of which

$$f(r) = 1 - \frac{(1 - \frac{1}{3}\Lambda r_h^2)r_h}{r} - \frac{\Lambda r^2}{3}. \quad (3)$$

Our interest in this elementary solution of GR comes from the fact that, contrary to what happens for $\Lambda \geq 0$ Schwarzschild black holes, for which a notion of nonmodal linear gravitational stability applies [4], SAdS₄ is unstable under many choices of boundary conditions, and the even-odd linear gravity duality is broken except for only two choices of boundary conditions, one of them leading to unstable dynamics (see section VII B 2 for details.)

The static region of SAdS₄ corresponds to $r_h < r < \infty$. There we define

$$x = - \int_r^\infty \frac{dr'}{f(r')} \quad (4)$$

and find that $x \in (-\infty, 0)$ with $x \rightarrow -\infty$ as $r \rightarrow r_h^+$ and $x \rightarrow 0$ as $r \rightarrow \infty$ in the following way:

$$x \simeq \begin{cases} \frac{r_h}{1-\Lambda r_h^2} \ln \left(\frac{r}{r_h} - 1 \right) & , r \rightarrow r_h^+ \\ \frac{3}{\Lambda r} & , r \rightarrow \infty. \end{cases} \quad (5)$$

In terms of x the metric on the static region reads

$$ds^2 = f(-dt^2 + dx^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

where $r = r(x)$ is the inverse of (4). The static region manifold $\mathbb{R}_t \times (-\infty, 0)_x \times S^2$ has, according to (6), the same causal structure of S^2 times *the* $x < 0$ half of Minkowski space in 1+1 dimensions. This is certainly not a globally hyperbolic spacetime: given any spacelike hypersurface Σ there will be causal lines not intersecting it (e.g., those which are future directed and originate at an $x = 0$ point to the future of Σ). As a consequence, fields obeying wave like equations are no longer determined by their values and time derivatives at, say, a $t = t_o$ surface Σ_{t_o} . The differential equation they satisfy has a unique solution only within the domain of dependence of Σ_{t_o} , and this is not the entire space. To assure uniqueness on the entire space, boundary conditions at the conformal timelike boundary $x = 0$ have to be specified. Different boundary conditions may be consistent with the field equations and yet they lead to different evolutions of the same initial datum.

The situation of SAdS₄ generalizes to the large class of asymptotically AdS static black hole solutions of $d = n + 2$ dimensional GR with horizon an Einstein manifold σ^n with metric \hat{g}_{AB} and Ricci tensor $\hat{R}_{AB} = (n - 1)\kappa\hat{g}_{AB}$, $\kappa = 0, \pm 1$. The metric of these black holes in static coordinates (t, r, z^A) is given by [17] [16]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\hat{g}_{AB}(z)dz^A dz^B, \quad (7)$$

with

$$f(r) = \kappa - \frac{2M}{r^{n-1}} - \frac{2\Lambda r^2}{n(n+1)}, \quad (8)$$

M the mass and Λ the cosmological constant. For $\Lambda < 0, M > 0$ and any κ , f grows monotonically for $r \in (0, \infty)$ from minus to plus infinity, with a simple zero at $r = r_h$ in terms of which

$$f = \kappa - \frac{\left(\kappa r_h^{(n-1)} - \frac{2\Lambda r_h^{n+1}}{n(n+1)}\right)}{r^{n+1}} - \frac{2\Lambda r^2}{n(n+1)}. \quad (9)$$

The static region corresponds to $r_h < r < \infty$, where we define x as in (4) and find that

$$x \simeq \begin{cases} \frac{1}{f'(r_h)} \ln\left(\frac{r}{r_h} - 1\right) & , r \rightarrow r_h^+ \\ \frac{n(n+1)}{2\Lambda r} & , r \rightarrow \infty. \end{cases} \quad (10)$$

In terms of x the static region metric is

$$ds^2 = f(-dt^2 + dx^2) + r^2\hat{g}_{AB}(z) dz^A dz^B. \quad (11)$$

As for SADS₄, $x \in (-\infty, 0)$, then the static region manifold $\mathbb{R}_t \times (-\infty, 0)_x \times \sigma^n$ has the causal structure of σ^n times *a half* of 1+1 Minkowski spacetime.

Alternative theories of gravity, such as Lovelock's, admit asymptotically (A)dS black hole solutions with metrics of the form (7) (see [18] and references therein). The function f is no longer given by (8), but in the asymptotically AdS case, by definition, $f \sim -\Lambda_{eff}r^2$ for large r and some negative effective cosmological constant Λ_{eff} , whereas near the event horizon $r = r_h$ (the largest positive root of f) $f \sim f'(r_h)(r - r_h)$, thus, the integral (4) that defines x (so that (11) holds) diverges logarithmically, and x is again restricted to a half line. This is very different to what happens in the asymptotically flat case, where $f \sim 1$ for large r , then $x \sim r$ in this limit and $x \in (-\infty, \infty)$. Similarly, in the asymptotically de Sitter case, the static region corresponds to $r_h < r < r_c$ (r_c is the cosmological horizon). Here r_h and r_c are simple roots of f that introduce logarithmic divergences in the integral defining x , therefore $x \in (-\infty, \infty)$. In any case the metric can be put in the form (11), but only in the asymptotically AdS case is x restricted to a half line and the static region is non globally hyperbolic.

The ambiguity that the non globally hyperbolic character of the static region of asymptotically AdS black holes may introduce in the dynamics of linear fields is the subject of this work.

For metrics of the form (7) with constant curvature horizons σ^n , scalar and Maxwell fields as well as linear metric perturbations can all be expanded as a series in a basis of eigentensors of the Laplace-Beltrami (LB) operator on σ^n , with “coefficients” that carry tensor indexes in the (t, r) Lorentzian *orbit manifold* [15, 16]. We call each term in this series a field *mode*. After some work, the (Maxwell, linear gravity, etc) field equations reduce in every case to an infinite set of 1+1 wave equations for a master variable, one for each mode, with a time independent potential. Since the massless wave equation in 1+1 dimension is conformally invariant, the master equation satisfied by the master variable has the Minkowskian form

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x) \right] \phi = 0. \quad (12)$$

For GR in arbitrary dimensions, this reduction was proved by Kodama and Ishibashi [15, 16]. For black holes with constant curvature horizons in the restricted case of second order Lovelock theories known as Einstein-Gauss-Bonnet gravity, the modal reduction of linear gravity to the form (12) was done in [19, 20]. Generalizations to higher order Lovelock theories can be found in [21]. Further examples of reduction of linear field equations to mode equations of the form (12) include hairy black holes, as in [8]. In all these cases, it is the warped product form of the background (7) what allows separation of variables, independently of the form of f . The horizon manifold drops out from the field equations, leaving a trace of it in the mode counting (modes are in one-one relation with the eigenspaces of the LB operator on different kinds of tensor fields on σ^n), and on the form of the potentials V for each mode. The non globally hyperbolic character of the spacetime is what implies the existence of a conformal *timelike* boundary at $x = 0$: contrary to what happens for $\Lambda \geq 0$, for which the domain of the wave equations (12) is 1+1 Minkowski spacetime, in the asymptotically AdS case $x < 0$. In view of these facts, the relevance to asymptotically anti de Sitter black holes of the study of equation (12) on a half Minkowski space can certainly not be overstated.

The purpose of this paper is to study equation (12) under different boundary conditions at the spacetime timelike conformal boundary at $x = 0$. It is a non trivial fact that, as far as we know, for all the above mentioned theories and fields (scalar, Maxwell and linear metric perturbations), either 1) V is continuous for $x \in (-\infty, 0]$ or 2) V is continuous in $(-\infty, 0)$ and diverges at the conformal boundary as $V \sim c/x^2$ for some constant c that depends on the field type and the mode. In all cases $V \rightarrow 0$ as $x \rightarrow -\infty$, i.e., as $r \rightarrow r_h^+$. We focus on the different dynamics that arise under homogeneous Dirichlet ($\phi|_{x=0} = 0$), Neumann ($\partial_x \phi|_{x=0} = 0$) or Robin ($\partial_x \phi|_{x=0} = \gamma \phi|_{x=0}$) boundary conditions, which are the natural ones, as they preserve the action of spacetime isometries on the solution space (see [2]), with an accent on the least studied Robin boundary condition, for which we find that there is always a range of the Robin parameter γ under which the dynamics is unstable. We will see, however, that this variety of boundary conditions is available only in case 1). In case 2) with $c < 3/4$ some generalized forms of Robin, Dirichlet and Neumann homogeneous boundary conditions can be defined, whereas for case 2) with $c \geq 3/4$ only homogeneous Dirichlet boundary conditions are allowed [3].

The approach in [1–3] to the issue of dynamics and the initial value problem on non globally hyperbolic spacetimes is reviewed briefly in Section II, and the modal decomposition, treated in full detail in [15, 16], is illustrated for SAdS₄ in Section III and for higher dimensional theories in Section V. In Section IV we classify these theories according to their admissible boundary conditions. Section VI contains our main results on equation (12) on half 1+1 Minkowski space for potentials of type 1). It is shown that for high enough Robin parameter there is an instability, and a number of Propositions are proven about the range of instabilities and the properties of the unstable modes. Section VII explores the effects of the Robin instability on SAdS₄ and analyzes the even-odd duality that is peculiar of dimension four. We show that this duality, which is reminiscent of supersymmetric Quantum Mechanics, is broken except for two out of the infinitely many choices of boundary conditions, only one of which gives a stable dynamics. A brief summary of our results can be found in Section VIII.

II. DYNAMICS IN STATIC NON GLOBALLY HYPERBOLIC SPACETIMES

The issue of dynamics in nonglobally hyperbolic, static space-times was studied in detail in [1] and the follow up papers [2] and [3]. A brief summary of the main results in [1] follows.

Consider the case of the wave equation with a potential $U(y)$,

$$\nabla_\alpha \nabla^\alpha \Phi - U\Phi = 0, \quad (13)$$

for a scalar field Φ in a static spacetime $\Sigma \times \mathbb{R}_t$,

$$ds^2 = -f(y)dt^2 + h_{ij}(y)dy^i dy^j, \quad f(y) > 0. \quad (14)$$

Let D_j be the covariant derivative of the Riemannian metric (Σ, h_{ij}) , $d\Sigma$ its volume form. Using (14) we find that

$$\nabla_\alpha \nabla^\alpha \Phi - U\Phi = -f^{-1} \left[\ddot{\Phi} + H\Phi \right], \quad (15)$$

where a dot denotes t derivative and

$$H\Phi = -f^{1/2} D_i \left(f^{1/2} D^i \Phi \right) + fU\Phi. \quad (16)$$

Under the inner product

$$\langle \Psi, \Phi \rangle = \int_\Sigma \Psi \Phi f^{-\frac{1}{2}} d\Sigma, \quad (17)$$

the operator H is symmetric, that is, if Ψ and Φ have compact support on Σ , then

$$\langle \Psi, H\Phi \rangle = \langle H\Psi, \Phi \rangle. \quad (18)$$

Consider an extension of the domain of H from compactly supported functions to some $\mathcal{O} \subset L^2(\Sigma, d\Sigma)$ chosen such that ${}^{\mathcal{O}}H$, the operator (16) acting on the domain \mathcal{O} , is self adjoint. Self-adjointness guarantees that ${}^{\mathcal{O}}H$ has a complete set of generalized eigenfunctions Φ_E , ${}^{\mathcal{O}}H\Phi_E = E\Phi_E$, meaning that any function Φ in \mathcal{O} can be written as

$$\Phi = \int dE c_E \Phi_E, \quad (19)$$

where $\int dE$ is shorthand for an integral over the continuous spectrum plus a sum over the discrete spectrum of generalized eigenfunctions of ${}^{\mathcal{O}}H$. Given $q = \Phi(t=0, y)$ and $p = \dot{\Phi}(t=0, y)$ both in \mathcal{O} ,

$$q = \int dE q_E \Phi_E, \quad (20)$$

$$p = \int dE p_E \Phi_E, \quad (21)$$

the solution of the scalar wave equation

$$\ddot{\Phi} + H\Phi = 0 \quad (22)$$

with initial data (p, q) is the curve in \mathcal{O} given by

$$t \rightarrow \Phi(t) = \int dE c_E(t) \Phi_E, \quad (23)$$

where $c_E(t)$ is the solution of the equation $\ddot{c}_E + E c_E = 0$ with initial conditions $c_E(0) = q_E, \dot{c}_E(0) = p_E$. Note that $c_E(t)$ will be oscillatory if $E > 0$, linear if $E = 0$ and exponential if $E < 0$. Thus, the field is unstable if ${}^{\mathcal{O}}H$ is not positive definite.

If H admits a unique self adjoint extension (this is the case when Σ is a Cauchy surface of the spacetime $\Sigma \times \mathbb{R}_t$), the evolution of initial data is unambiguous. If not, different self adjoint extensions give different $\Phi(t)$ which, in view of the standard results on hyperbolic equations, must agree in the domain of dependence of the initial data surface. Solutions of the wave equation using different self adjoint extensions of H will disagree only outside this domain. In any case, if the initial data is smooth and compactly supported, (23) defines for every t a smooth function $\Phi(t, \cdot)$ on Σ [1].

The way that boundary conditions enter the problem is in the definition of the subspace $\mathcal{O} \subset L^2(\Sigma, d\Sigma)$. Boundary conditions have to be consistent with the differential equation, and will depend crucially on the behavior of U near the boundary. Equation (23) implies that a consistent choice of boundary condition is preserved by the evolution, as the right hand side of (23) belongs to \mathcal{O} for every t .

III. MODE DECOMPOSITION OF LINEAR FIELDS ON SADS₄

In this Section we illustrate the modal decomposition for the Klein Gordon and Maxwell fields and for linear metric perturbations on the SAdS₄ background (see, e.g., [5]). For a systematic treatment of the modal decomposition on dimension $D \geq 4$ black holes with constant curvature horizon manifolds we refer the reader to [15, 16]. The modal decomposition reduces the problem of linear fields propagating on a background (11) to a set of 1+1 wave equations of the form (12), independently of the dimension of the background.

For SAdS₄ we find that for the scalar field there is a unique possible boundary condition at the time-like boundary and that the field is stable, whereas Maxwell fields and gravitational perturbations allow infinitely many different boundary conditions. Higher dimensional backgrounds will be considered in Section V.

A. Massless scalar fields on SAdS₄

For the SAdS₄ metric, (15) simplifies to

$$\nabla_\alpha \nabla^\alpha \Phi - U\Phi = (rf)^{-1} \left[-\partial_t^2 + f\partial_r(f\partial_r) + f(r^{-2}\Delta - f'/r - U) \right] (r\Phi), \quad (24)$$

where Δ is the Laplacian on the unit sphere

$$\Delta = \partial_\theta^2 + \cot(\theta) \partial_\theta + \sin(\theta)^{-2} \partial_\varphi^2. \quad (25)$$

Introducing

$$\phi = r\Phi, \quad rH\Phi = \mathcal{H}(r\Phi) = \mathcal{H}\phi \quad (26)$$

in (24), equation (15) is seen to be equivalent to

$$-\ddot{\phi} = \mathcal{H}\phi, \quad (27)$$

where

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + V_1 - V_2 \Delta, \quad V_1 = f(U + f'/r), \quad V_2 = f/r^2, \quad (28)$$

and x was defined in (4). The inner product (17) is equivalent to

$$\langle \Psi, \Phi \rangle = (\psi, \phi) := \int \psi \phi \, d\sigma, \quad d\sigma = \sin(\theta) \, dx \, d\theta \, d\varphi = r^{-2} f^{-\frac{1}{2}} d\Sigma, \quad (29)$$

and

$$\langle \Psi, H\Phi \rangle = (\psi, \mathcal{H}\phi), \quad (30)$$

with \mathcal{H} symmetric on compactly supported functions as a consequence of (18): $(\psi, \mathcal{H}\phi) = \langle \Psi, H\Phi \rangle = \langle H\Psi, \Phi \rangle = (\mathcal{H}\psi, \phi)$. If we expand ϕ using an $L^2(S^2)$ orthonormal basis $S_{(\ell,m)}$ of real spherical harmonics on the sphere,

$$\phi = \sum_{(\ell,m)} r^{-1}(x) \phi_{(\ell,m)}(t, x) S_{(\ell,m)}(\theta, \phi), \quad (31)$$

we find from (29) that the inner product (17) is equivalent to

$$\langle \Psi, \Phi \rangle = (\psi, \phi) := \sum_{(\ell,m)} \int_{-\infty}^0 \psi_{(\ell,m)} \phi_{(\ell,m)} dx, \quad (32)$$

and that (27) is equivalent to a set of wave equations in the $x < 0$ half of 1+1 Minkowski spacetime:

$$-\ddot{\phi}_{(\ell,m)} = \mathcal{H}_\ell \phi_{(\ell,m)}, \quad (33)$$

with

$$\mathcal{H}_\ell = -\frac{\partial^2}{\partial x^2} + V_\ell, \quad (34)$$

and $V_\ell = V_1 + \ell(\ell+1)V_2$ (c.f. equation (28)). Choosing a boundary condition at $x = 0$ consistent with (33) defines a self adjoint extension of \mathcal{H}_ℓ within $L^2((-\infty, 0), dx)$. In view of (29) and (32), the resulting set of boundary conditions at $x = 0$ (one for each ℓ) defines a boundary condition –and therefore a self adjoint extension– of \mathcal{H} within $L^2(\Sigma, d\sigma)$ (equivalently, H within $L^2(\Sigma, f^{-1/2}d\Sigma)$), Σ a $t = \text{constant}$ hypersurface. If for some ℓ the chosen extension contains a negative eigenvalue in its spectrum, this mode will grow unbounded in time, and $\Phi = \sum_{(\ell,m)} \phi_{(\ell,m)} S_{(\ell,m)}/r$ will be unstable.

Massless scalar fields on SAdS₄ obey equation (24) with $U = 0$, which is equivalent to the set of equations (33)-(34) with

$$V_\ell^s = f \left[\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2} - \frac{2}{3}\Lambda \right]. \quad (35)$$

Note that $V_\ell^s > 0$ and that

$$V_\ell^s \sim \begin{cases} \frac{2}{x^2} - \frac{\Lambda}{3}(\ell^2 + \ell + 2) + \mathcal{O}(x) & , \text{ as } x \rightarrow 0^- \\ (\ell(\ell+1) + 1 - \Lambda r_h^2)(1 - \Lambda r_h^2) r_h^{-2} \exp\left(\left(\frac{1-\Lambda r_h^2}{r_h}\right)x\right) & , \text{ as } x \rightarrow -\infty \end{cases} \quad (36)$$

According to Theorem X.10 in [7], the fact that the x^{-2} coefficient in (36) is greater than 3/4 implies that the potential V_ℓ^s belongs to the *limit point case* at $x = 0$ (see Section IV), which means that the space of local solutions of

$$(-\partial_x^2 + V_\ell^s)\psi = E\psi \quad (37)$$

for which $\int_c^0 \psi^2 dx < \infty$ for some negative c is one dimensional. Indeed, the two dimensional space of Frobenius series solutions for (37) at $x = 0$ ($r = \infty$) is

$$\psi = A \left(\frac{1}{r^2} - \frac{3}{10\Lambda^2} \frac{(\ell+3)(\ell-2)\Lambda + 3E}{r^4} + \mathcal{O}(r^{-5}) \right) + B \left(r + \frac{3}{2\Lambda^2} \frac{\ell(\ell+1)\Lambda + 3E}{r} + \mathcal{O}(r^{-2}) \right), \quad (38)$$

and $B = 0$ is required for the integral $\int_{r_o}^\infty \psi^2 dr/f$ to converge.

As $x \rightarrow -\infty$ ($r \rightarrow r_h^+$),

$$\psi = C(r - r_h)^\kappa (1 + \mathcal{O}(r - r_h)) + D(r - r_h)^{-\kappa} (1 + \mathcal{O}(r - r_h)), \quad \kappa = \frac{\sqrt{-E} r_h^2}{1 - \Lambda r_h^2}, \quad (39)$$

where it is assumed that we take real part. If $E > 0$, the solution of (37) that is square integrable near $x = 0$ (the one with $B = 0$ in (38)) will behave near the horizon as in (39). This oscillatory behavior is characteristic of generalized eigenfunctions for potentials that vanish in this limit, equation (36).

A negative value of E (real positive κ) would be admissible if there were solutions of (37) behaving as in (38) with $B = 0$ near $x = 0$ while behaving as in (39) with $D = 0$ near the horizon. However, if we assume $B = D = 0$, and $E < 0$, then $\int_{-\infty}^0 \psi^2 dx$ would converge and we arrive at a contradiction:

$$\begin{aligned} E \int_{-\infty}^0 \psi^2 dx &= \int_{-\infty}^0 \psi (-\partial_x^2 \psi + V_\ell^s \psi) dx = \\ &= [\psi f \partial_r \psi] \Big|_{r=r_h}^{r=\infty} + \int_{-\infty}^0 ((\partial_x \psi)^2 + V_\ell^s \psi^2) dx = \int_{-\infty}^0 ((\partial_x \psi)^2 + V_\ell^s \psi^2) dx > 0. \end{aligned} \quad (40)$$

We conclude that the dynamics of massless scalar fields on SAdS is not ambiguous, as there is a unique possible admissible condition at $x = 0$, namely, $B = 0$ in (38), i.e., Dirichlet. The extension of the domain of $H_\ell^s = -\partial_x^2 + V_\ell^s$ from functions of compact support to allow $\psi \sim r^{-2}$ for large r gives a self adjoint operator which, as we have just

shown, has a positive spectrum. As a consequence, no modal instability, that is, isolated modes growing linearly or exponentially in time, exists.

Maxwell fields, as well as gravitational perturbation, exhibit more complicated patterns, with infinitely many possible boundary conditions, some of them leading to unstable dynamics. To show this, we need recall how to reduce Maxwell equations and the linearized Einstein equations (LEE) to a set of the form (33)-(34) using tensor decompositions into harmonic S^2 tensors, generalizing what was done in (31) for the scalar field. This is a well known procedure which, for $D = 4$, is reviewed, e.g., in [4], [5] and specifically for SAdS₄ in [6].

B. Maxwell fields on SAdS₄

Write the Maxwell potential as a sum of its vector/odd (−) and scalar/even (+) pieces, $A_\beta = A_\beta^{(-)} + A_\beta^{(+)}$ [4–6], these can be gauge fixed to the form

$$A_\beta^{(-)} = \sum_{(\ell, m)} \phi^{(-, \ell, m)} (0, 0, \frac{1}{\sin \theta} \partial_\phi S_{(\ell, m)}, -\sin \theta \partial_\theta S_{(\ell, m)}) \quad (41)$$

$$A_\beta^{(+)} = \sum_{(\ell, m)} (f \partial_r \phi^{(+, \ell, m)} S_{(\ell, m)}, f^{-1} \partial_t \phi^{(+, \ell, m)} S_{(\ell, m)}, 0, 0). \quad (42)$$

where $\phi^{(\pm, \ell, m)}$ are functions of (t, r) . Maxwell equations $F = dA$ and $\nabla^\alpha F_{\alpha\beta} = 0$ are equivalent to

$$-\ddot{\phi}_{(\ell, m)}^\pm = \mathcal{H}_{(\ell, m)}^{Max} \phi_{(\ell, m)}^\pm, \quad (43)$$

where the Hamiltonian

$$\mathcal{H}_{(\ell, m)}^{Max} = -\partial_x^2 + V_\ell^{Max} \quad (44)$$

is independent of the \pm parity and the azimuthal number m and has a potential

$$V_\ell^{Max} = f \frac{\ell(\ell+1)}{r^2}. \quad (45)$$

Note that $V_\ell^{Max} > 0$ and that

$$V_\ell^{Max} \sim \begin{cases} -\frac{\Lambda}{3} \ell(\ell+1) + \mathcal{O}(x^2) & , \text{ as } x \rightarrow 0^- \\ \ell(\ell+1)(1 - \Lambda r_h^2) r_h^{-2} \exp\left(\left(\frac{1 - \Lambda r_h^2}{r_h}\right)x\right) & , \text{ as } x \rightarrow -\infty \end{cases} \quad (46)$$

For future reference we also note that

$$\int_{-\infty}^0 V_\ell^{Max} dx = \int_{r_h}^\infty V_\ell^{Max} \frac{dr}{f} = \frac{\ell(\ell+1)}{r_h}. \quad (47)$$

C. Gravitational waves on SAdS₄

As done for Maxwell fields, we can decompose metric perturbations into scalar/even/+ and vector/odd/- fields with harmonic numbers (ℓ, m) . This procedure is well known, the details can be found in [4, 5] and references therein. The $\ell = 0, 1$ sectors contain either pure gauge fields, or time independent fields which corresponds to perturbations within the Kerr family, that is, a variation of mass, or an addition of angular momentum. These are irrelevant to the stability problem, so we will focus on the $\ell \geq 2$ modes.

For $\pm, \ell \geq 2$ modes the LEE reduce to the well known Regge-Wheeler (−) and Zerilli (+) equations for the gauge invariant fields $\phi_{(\ell, m)}^\pm(t, r)$, which are of the form (33)-(34). The Regge-Wheeler potential of the odd Hamiltonian \mathcal{H}_ℓ^- is

$$V_\ell^{(-)} = f \left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} \right), \quad (48)$$

and is positive in the outer static region $r > r_h$ except when r_h is small compared to M , case in which is negative in the interval $r_h < r < 6M/(\ell(\ell+1))$.

The Zerilli potential of the even Hamiltonian \mathcal{H}_ℓ^+ is

$$V_\ell^{(+)} = f \frac{[\mu^2 \ell(\ell+1) - 24M^2 \Lambda] r^3 + 6\mu^2 M r^2 + 36\mu M^2 r + 72M^3}{r^3 (6M + \mu^2 r)^2}, \quad \mu = (\ell-1)(\ell+2), \quad (49)$$

this potential is positive for $r > r_h$ and behaves as

$$V_\ell^{(+)} \sim \begin{cases} \frac{24M^2 \Lambda^2}{\mu^2} - \Lambda \ell(\ell+1) + \mathcal{O}(x) & , \text{ as } x \rightarrow 0^- \\ \frac{(\Lambda^2 r_h^4 - 4\Lambda r_h^2 + \ell^4 + 2\ell^3 - \ell^2 - 2\ell + 3)(1 - \Lambda r_h^2)}{(1 - \Lambda r_h^2 + \ell(\ell+1)) r_h^2} \exp\left(\left(\frac{1 - \Lambda r_h^2}{r_h}\right)x\right) & , \text{ as } x \rightarrow -\infty \end{cases} \quad (50)$$

For future reference we note that

$$\int_{-\infty}^0 V_\ell^{(+)} dx = \frac{2\Lambda^2 r_h^3}{3(\ell+2)(\ell-1)} + \frac{\Lambda r_h(\ell+3)(\ell-2)}{2(\ell+2)(\ell-1)} + \frac{2\ell^2 + 2\ell - 3}{2r_h} \quad (51)$$

IV. BOUNDARY CONDITIONS AND SELF-ADJOINT EXTENSIONS

For scalar and Maxwell fields, as well as linear metric perturbations, modal decompositions such as those illustrated in the previous Section for SAdS₄ are possible also on higher dimensional black holes with a constant curvature manifold as a horizon. This was proved by Kodama and Ishibashi, see e.g., [15, 16]. For Maxwell fields, Kodama and Ishibashi find the analogue of the odd/vector and even/scalar modes of Section III B. For linear metric perturbations besides there is a third family of *tensor* modes.

After expanding in modes the field equations reduce to a set of equations (one for each mode) of the form:

$$-\ddot{\phi} = \mathcal{H} \phi, \quad (52)$$

where the Hamiltonian operator is

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + V(x), \quad (53)$$

with V continuous in $(-\infty, 0)$, $V(x) \rightarrow 0$ as $x \rightarrow -\infty$. Examples are given for higher dimensional GR and Lovelock theories in Section V.

As with equation (22), the solution of (52) is of the form

$$\phi = \int dE c_E(t) \psi_E(x), \quad \ddot{c}_E + E c_E = 0, \quad (54)$$

where the ψ_E are generalized eigenfunctions

$${}^X \mathcal{H} \psi_E = E \psi_E \quad (55)$$

of a chosen self adjoint extension ${}^X \mathcal{H}$ with domain a linear subset of $L^2((-\infty, 0), dx)$. In what follows we will use \mathcal{H} to denote the operator (53) acting on unspecified functions, and particular self adjoint extensions to specified domains in $L^2((-\infty, 0), dx)$ with a left upper superscript.

A negative energy $E < 0$ in the spectrum of the chosen extension allows exponentially growing terms in (54). Thus, there will be an instability whenever the spectrum of ${}^X \mathcal{H}$ contains a negative E value. The possible self adjoint extensions depend on the behavior of V near $x = 0$ and minus infinity. The potential V is said to be *limit circle case* (LC) at $x = 0$ if any function in the two dimensional space of solutions of the differential equation (55) is square integrable near zero, and is otherwise said to be *limit point case* (LP) at $x = 0$. The same notion applies at minus infinity (see the appendix to section X.I in [7]). It is a non trivial fact that, according to Theorem X.6 b in [7], the LC/LP notion does not depend on the value of E in (55) as long as V is continuous for $x \in (-\infty, 0)$.

The potentials that appear in the modal decompositions of linear fields are always LP at $x = -\infty$. Some of them are LC at $x = 0$, and some others are LP at $x = 0$. In the first case, *any* solution of (55) is of the form $\psi = A\psi_1 + B\psi_2$, where both ψ_1 and ψ_2 are square integrable near $x = 0$. Writing $(A, B) = C(\cos(\alpha), \sin(\alpha))$ and allowing the irrelevant overall factor C to be positive or negative while restricting $\alpha \in (-\pi/2, \pi/2]$, we find that α parametrizes the set of allowed boundary conditions at $x = 0$. If ${}^\alpha\mathcal{H}$ is the operator (53) with domain the linear subset of $L^2((-\infty, 0), dx)$ of functions with boundary condition at $x = 0$ consistent with choosing α above, then ${}^\alpha\mathcal{H}$ is self adjoint. Any self adjoint extension of \mathcal{H} is of this form. Thus, if V is LC at $x = 0$, there are infinitely many self adjoint extensions, and the dynamics is unambiguous only after selecting a self adjoint extension α . On the other hand, if V is LP at $x = 0$ there is no such ambiguity, as there is a single choice of boundary condition under which \mathcal{H} is self adjoint. In this case we have unique evolution of initial data at a t slice in spite of the non globally hyperbolic character of the outer static region. This is ultimately due to the fact that we have constrained boundary conditions at $x = 0$ to those making \mathcal{H} self-adjoint. In principle, there is a much wider set of possible boundary conditions. However, if we require that the time translation symmetry acts in the expected way on the space of solutions and that there is a conserved energy, we are forced to adopt solutions of the form (54) for some self adjoint extension of \mathcal{H} [2]. Thus, the possibilities for linear fields evolving on the outer static region of AdS black holes are that either there is a unique dynamics (corresponding to the case where each V is LP at $x = 0$) or a set of dynamics parametrized by $\alpha \in (-\pi/2, \pi/2]$ for each mode such that V is LC at $x = 0$.

In the theories we examined, the behavior of the potentials near the horizon is

$$V = f(r)[U_o + \mathcal{O}(r - r_h)], \quad U_o \neq 0, \quad r \simeq r_h. \quad (56)$$

In this limit

$$x \simeq \frac{\ln\left(\frac{r}{r_h} - 1\right)}{f'(r_h)}, \quad (57)$$

therefore (56) gives an exponential decay of V as $x \rightarrow -\infty$ (note that $f'(r_h) > 0$):

$$V \simeq f'(r_h) r_h e^{f'(r_h) x} [U_o + \dots] \quad (58)$$

The behavior of these potentials as $r \rightarrow \infty$ allowed us to classify them into two types:

Type 1: V is continuous for $x \in (-\infty, 0]$, satisfies (56) and for $x \simeq 0$

$$V(x) = v_0 + v_1 x + v_2 x^2 + \dots, \quad (v_0 \neq 0). \quad (59)$$

Type 2: V is continuous for $x \in (-\infty, 0)$, satisfies (56) and for $x \simeq 0$ diverges as x^{-2} :

$$V(x) = v_{-2} x^{-2} + v_{-1} x^{-1} + v_0 + v_1 x + v_2 x^2 + \dots, \quad (v_{-2} \neq 0). \quad (60)$$

The local Frobenius series solution of the differential equation (55) near $r = r_h$ is, using (56)

$$\psi_E = C \cos(\beta) (r - r_h)^{\frac{\sqrt{-E}}{f'(r_h)}} [1 + \mathcal{O}(r - r_h)] + C \sin(\beta) (r - r_h)^{-\frac{\sqrt{-E}}{f'(r_h)}} [1 + \mathcal{O}(r - r_h)]. \quad (61)$$

From (57),

$$(r - r_h)^{\pm \frac{\sqrt{-E}}{f'(r_h)}} \simeq r_h^{\pm \frac{\sqrt{-E}}{f'(r_h)}} e^{\pm \sqrt{-E} x}, \quad (62)$$

therefore for positive E (61) gives oscillatory solutions, which is a suitable behavior for generalized eigenfunctions, whereas for negative E only the $\beta = 0$ solution, which decays exponentially with x as $x \rightarrow -\infty$, would be allowed.

When analyzing the local solutions of (55) near $x = 0$ for potentials of types 1 and 2 above, we find that v_0 (and therefore E) appears only in sub-leading terms. Thus, any statement of local integrability near $x = 0$ is independent of E , as anticipated by Theorem X.6 b in [7]. However, there are crucial differences for type 2 potentials depending on the value of v_{-2} . This leads to the following finer classification (note the similarities with the classification in [3] for fields on AdS spacetime):

Type 1 : The local solution of (55) with potential (59) near $x = 0$ is

$$\psi_E = C \cos(\alpha) \left[1 + \left(\frac{v_o - E}{2} \right) x^2 + \frac{v_1}{6} x^3 + \mathcal{O}(x^4) \right] + C \sin(\alpha) \left[x - \left(\frac{v_o - E}{6} \right) x^3 + \mathcal{O}(x^4) \right], \quad (63)$$

where $A = C \cos(\alpha) = \psi_E(0)$ and $B = C \sin(\alpha) = \partial_x \psi_E(0)$, $\alpha \in (-\pi/2, \pi/2]$ and we allow negative values of C . The possible boundary conditions are Dirichlet ($\alpha = \pi/2$), Neumann ($\alpha = 0$) or Robin (the remaining cases). Robin boundary conditions are characterized by the nonzero value of $\psi'_E(0)/\psi_E(0) = \tan(\alpha) \equiv \gamma$.

Type 2.1 : $v_{-2} < -1/4$. Write $v_{-2} = -\nu^2 - 1/4$, $\nu > 0$. The local solution of (55) with potential (60) near $x = 0$ is

$$\psi_E = C \cos(\alpha) x^{\frac{1}{2} + i\nu} \left[1 + \left(\frac{v_{-1}}{2i\nu + 1} \right) x + \mathcal{O}(x^2) \right] + C \sin(\alpha) x^{\frac{1}{2} - i\nu} \left[1 - \left(\frac{v_{-1}}{2i\nu + 1} \right) x + \mathcal{O}(x^2) \right]. \quad (64)$$

This is a LC case since generic solutions are square integrable near $x = 0$.

Type 2.2 : $v_{-2} = -1/4$, the local solution near $x = 0$ is

$$\psi_E = C \cos(\alpha) |x|^{1/2} [1 + v_{-1}x + \mathcal{O}(x^2)] + C \sin(\alpha) |x|^{1/2} [\ln(x) + v_{-1}(\ln(x) - 2)x + \mathcal{O}(\ln(x)x^2)], \quad (65)$$

This is a LC case, generic solutions are square integrable near $x = 0$.

Type 2.3 : $-1/4 < v_{-2} < 3/4$, $v_{-2} \neq 0$. Write $v_{-2} = \nu^2 - 1/4$, where $0 < \nu < 1$, $\nu \neq 1/2$. The local solution near $x = 0$ is

$$\psi_E = C \cos(\alpha) x^{\frac{1}{2} + \nu} \left[1 + \left(\frac{v_{-1}}{2\nu + 1} \right) x + \mathcal{O}(x^2) \right] + C \sin(\alpha) x^{\frac{1}{2} - \nu} \left[1 - \left(\frac{v_{-1}}{2\nu - 1} \right) x + \mathcal{O}(x^2) \right], \quad (66)$$

This is a LC case, generic solutions are square integrable near $x = 0$.

Type 2.4 : $v_{-2} \geq 3/4$. Write $v_{-2} = \nu^2 - 1/4$, $\nu \geq 1$. The local solution near $x = 0$ is

$$\psi_E = C \cos(\alpha) x^{\frac{1}{2} + \nu} \left[1 + \left(\frac{v_{-1}}{2\nu + 1} \right) x + \mathcal{O}(x^2) \right] + C \sin(\alpha) x^{\frac{1}{2} - \nu} \left[1 - \left(\frac{v_{-1}}{2\nu - 1} \right) x + \mathcal{O}(x^2) \right]. \quad (67)$$

Only $\alpha = 0$ solutions are square integrable, this is an LP case. The allowed eigenfunctions satisfy Dirichlet boundary conditions at $x = 0$.

In the following Section, Maxwell fields and linearized gravity around asymptotically AdS black holes in higher dimensional GR and Lovelock theories are classified following this scheme.

Note that, although the field and its derivative are both defined at $x = 0$ in the type 1 case only, some generalized forms of Dirichlet, Robin and Neumann boundary conditions can be defined for type 2 potentials (see, e.g., [3]).

V. HIGHER DIMENSIONAL ASYMPTOTICALLY ADS BLACK HOLES

In this Section we consider Maxwell fields on higher dimensional asymptotically anti de Sitter black hole solutions of GR, and linearized gravity around these solutions and around asymptotically AdS black holes in second order Lovelock gravity, known as Einstein-Gauss-Bonnet (EGB) theories. The potentials for the master variables in the modal decompositions, obtained from references [11] [17] [15] [20] [19] [21], are used to classify these linear fields according to the scheme developed in the previous Section.

A. Higher dimensional GR

We consider Maxwell fields on the metric (7)-(8) and the linearized GR equations around this metric. The effective potentials for the Maxwell field can be obtained from reference [16] by setting $Q = 0$ (Q is the charge of the black hole). The electromagnetic master variables decouple from the gravitational ones for $Q = 0$, the resulting scalar and vector electromagnetic perturbations correspond respectively to the \pm modes studied in Section III B for $d = 4$.

As anticipated, for gravitational perturbations in dimension $d = 2 + n > 4$, besides there is a third family of *tensor* perturbations.

1. Electromagnetic perturbations

Vector and scalar types of Maxwell fields are respectively constructed using LB eigen-vector/scalar fields on the horizon manifold σ^n . We introduce $\lambda = \frac{2\Lambda}{n(n+1)}$ where $n = d-2$, d the spacetime dimension, and note that $x \sim -(\lambda r)^{-1}$ for large r .

1. The potential for vector type Maxwell perturbations, equation (5.23) in [16], is

$$V_V^{(em)} = \frac{f}{r^2} \left[k_V^2 + \kappa \frac{n^2 - 2n + 4}{4} - \frac{n(n-2)}{4} \lambda r^2 + \frac{(n^2 - 4)}{2} \frac{M}{r^{n-1}} \right], \quad (68)$$

where k_V^2 is the eigenvalue of the LB operator on vector fields on σ^n corresponding to the mode. In the case $\sigma^n = S^n$ the spectrum of the LB operator on vector fields is given by $k_V^2 = \ell(\ell + n - 1) - 1$, then for spherical four dimensional black holes ($n = 2$ and $\kappa = 1$) we find that (68) reduces to (45), as expected. For $n > 2$, the behavior of (68) near the conformal boundary is

$$V_V^{(em)} \sim \frac{n(n-2)}{4} \lambda^2 r^2 \sim \frac{n(n-2)}{4} x^{-2}, \quad x \rightarrow 0^-, \quad n > 2. \quad (69)$$

According to the classification of Section IV, $\frac{n(n-2)}{4} \geq \frac{3}{4}$ (for $n > 2$) implies that $V_V^{(em)}$ is type 2.4. For $n = 2$ (68) gives

$$V_V^{(em)} \sim -\lambda(k_V^2 + \kappa), \quad x \rightarrow 0^-, \quad n = 2, \quad (70)$$

and then $V_V^{(em)}$ for $n = 2$ is type 1.

2. The potential for scalar type Maxwell perturbations, equation (6.23) in [16], is

$$V_S^{(em)} = \frac{f}{4r^2} \left[-(n-2)(n-4)\lambda r^2 - (3n-2)(n-2)\frac{2M}{r^{n-1}} + 4(k_S^2 - n\kappa) + n(n+2)\kappa \right], \quad (71)$$

where k_S^2 is the eigenvalue of the LB operator on scalar fields on σ^n corresponding to the mode. In the case $\sigma^n = S^n$ the spectrum of the LB operator on scalar fields is given by $k_S^2 = \ell(\ell + n - 1)$, then for spherical horizon four dimensional black holes ($n = 2$ and $\kappa = 1$) we find that (68) also reduces to the expected result (45). For $n \neq 2, 4$, the behavior of (68) near the conformal boundary is

$$V_S^{(em)} \sim \frac{(n-2)(n-4)}{4} \lambda^2 r^2 \sim \frac{(n-2)(n-4)}{4} x^{-2}, \quad x \rightarrow 0^-, \quad n \neq 2, 4. \quad (72)$$

For $n = 3$, $V_S^{(em)} \sim -\frac{1}{4}x^{-2}$ implies that the potential is type 2.2 according to the scheme of section IV. For $n \geq 5$ we see that $V_S^{(em)}$ is type 2.4. For the exceptional cases $n = 2, 4$,

$$V_S^{(em)} \sim -\lambda \left[(k_S^2 - n\kappa) + \frac{n(n+2)\kappa}{4} \right], \quad x \rightarrow 0^-, \quad n = 2, 4 \quad (73)$$

and then the potential is type 1 in these cases.

2. Gravitational perturbations

Metric perturbations are classified into three different types: tensor, vector and scalar. These are respectively constructed using LB eigen-tensor/vector/scalar fields on the horizon manifold σ^n . For $d = 4$ spherical black holes the vector and scalar perturbations correspond respectively to the odd and even modes treated in Section III C and there are no tensor perturbations.

1. The potential for tensor type gravitational perturbations, equation (2.10) in [23], is

$$V_T = \frac{f}{r^2} \left[\frac{n(n+2)}{4} f + \frac{n(n+1)M}{r^{n-1}} + k_T^2 - (n-2)\kappa \right], \quad (74)$$

where k_T^2 is the eigenvalue of the LB operator on symmetric divergence free rank two tensor fields on σ^n corresponding to the mode [23]. In the case $\sigma^n = S^n$, we have $k_T^2 := \ell(\ell + n - 1) - 2$. V_T is positive definite and near the conformal boundary behaves as

$$V_T \sim \frac{n(n+2)}{4} \lambda^2 r^2 \sim \frac{n(n+2)}{4} x^{-2}, \quad x \rightarrow 0^-. \quad (75)$$

Since $\frac{n(n+2)}{4} > \frac{3}{4}$ for $n > 2$, V_T is type 2.4 according to the classification of Section IV (note that tensor mode perturbations exist only for $n > 2$).

2. The potential for vector type gravitational perturbations, equation (2.17) in [23], is

$$V_V = \frac{f}{r^2} \left[\frac{n(n+2)}{4} f - \frac{nr}{2} f' + k_V^2 - (n-1)\kappa \right], \quad (76)$$

This potential is generally not positive-definite, and it reduces to the Regge-Wheeler form for spherical four dimensional black holes. For $n > 2$ the behavior of V_V near the conformal boundary is

$$V_V \sim \frac{n(n-2)}{4} \lambda^2 r^2 \sim \frac{n(n-2)}{4} x^{-2}, \quad x \rightarrow 0^-, \quad n > 2. \quad (77)$$

Similarly to tensor modes, $\frac{n(n-2)}{4} > \frac{3}{4}$ for $n > 2$ implies that V_V is type 2.4 in these cases. For the exceptional case $n = 2$,

$$V_V \sim -\lambda(k_V^2 + \kappa) \quad x \rightarrow 0^-, \quad n = 2. \quad (78)$$

V_V in $n = 2$ is then regular for $x \in (-\infty, 0]$, and it is of type 1 according to the classification of section IV.

3. The potential for scalar type gravitational perturbations, equation (2.31) in [23], is [?]]

$$V_S = \frac{f}{4r^2} \frac{r^{2(n-1)} Q(r)}{[2mr^{n-1} + 2Mn(n+1)]^2}, \quad (79)$$

where $m = k_S^2 - n\kappa$ and

$$\begin{aligned} Q(r) = & - \left[n^3(n+2)(n+1)^2 \frac{4M^2}{r^{2(n-2)}} - 12n^2(n+1)(n-2)m \frac{2M}{r^{n-3}} + 4(n-2)(n-4)m^2 r^2 \right] \lambda \\ & + n^4(n+1)^2 \frac{8M^3}{r^{3(n-1)}} + n(n+1) [4(2n^2 - 3n + 4)m + n(n-2)(n-4)(n+1)\kappa] \frac{4M^2}{r^{2(n-1)}} \\ & - 12nm [(n-4)m + n(n+1)(n-2)] \frac{2M}{r^{n-1}} + 16m^3 + 4n(n+2)m^2 \kappa. \end{aligned} \quad (80)$$

It can be shown that, as expected, V_S reduces to the Zerilli potential (49) for $n = 2, \kappa = 1$. In this case it satisfies $V_S \geq 0$, but more generally it is not positive definite for higher dimensions $n \geq 3$. For $n \neq 2, 4$, the behavior of (79) near the conformal boundary is

$$V_S \sim \frac{(n-2)(n-4)}{4} \lambda^2 r^2 \sim \frac{(n-2)(n-4)}{4} x^{-2}, \quad r \rightarrow \infty. \quad (81)$$

For $n = 3$, we get $V_S \sim -\frac{1}{4}x^{-2}$, which implies that V_S is type 2.2 according to the classification of section IV, while for $n \geq 5$ we see that V_S is type 2.4. For the exceptional cases $n = 2, 4$,

$$V_S \sim -\lambda \left[(k_S^2 - n\kappa) + \frac{n(n+2)\kappa}{4} \right], \quad x \rightarrow 0^-, \quad n = 2, 4 \quad (82)$$

This behavior implies that V_S is type 1 for $n = 2, 4$.

B. Einstein-Gauss-Bonnet theory

We follow the conventions and notation of [19, 20]. The field equations are $\Lambda G_{\alpha\beta}^{(0)} + G_{\alpha\beta}^{(1)} + \alpha G_{\alpha\beta}^{(2)} = 0$, where $G_{\alpha\beta}^{(0)} = g_{\alpha\beta}$, $G_{\alpha\beta}^{(1)}$ is the Einstein tensor and $G_{\alpha\beta}^{(2)}$ the quadratic Gauss-Bonnet tensor. In this theory there are static, $d = (2+n)$ -dimensional black hole solutions of the form (7), where now $f(r)$ is given by

$$f(r) = \kappa - r^2\psi(r), \quad \kappa = 0, \pm 1 \quad (83)$$

and $\psi(r)$ satisfies

$$\alpha P(\psi(r)) \equiv \frac{\alpha n(n-1)(n-2)}{4} \psi(r)^2 + \frac{n}{2} \psi(r) - \frac{\Lambda}{n+1} \equiv \frac{\mu}{r^{n+1}}. \quad (84)$$

We will assume that the function $P(\psi)$ has two *real* roots Λ_{\pm} , since in this case r extends to infinity. We see then that $\psi(r)$ behaves asymptotically as $\psi(r) \sim \Lambda_{\pm}$, $r \rightarrow \infty$, where the roots Λ_{\pm} are effective cosmological constants given by

$$\Lambda_{\pm} := \frac{1}{\alpha(n-1)(n-2)} \left[-1 \pm \sqrt{1 + \frac{4\alpha\Lambda(n-1)(n-2)}{n(n+1)}} \right]. \quad (85)$$

The asymptotic form of $f(r)$ is $f(r) \sim -\Lambda_{\pm} r^2$, $r \rightarrow \infty$. As usual, we define a tortoise coordinate x as $dx = dr/f$, whose asymptotic behavior is

$$x \sim \frac{1}{\Lambda_{\pm} r}, \quad r \rightarrow \infty. \quad (86)$$

Gravitational perturbations of these spacetimes are described by wave equations like (12), where again we analyze separately the different sectors (tensor, vector, scalar) of the perturbation. Since the scalar sector involves rather long and complicated expressions [20], we will restrict our analysis to the tensor and vector sectors:

1. The potential for the tensor sector of perturbations is

$$V_T(r) = q_T(r) + \frac{f}{K_T} \frac{d}{dr} \left(f \frac{dK_T}{dr} \right), \quad (87)$$

where

$$q_T(r) := (2\kappa - k_T^2) \frac{f}{r^2} \left[\frac{(1 - \alpha f'')r^2 + \alpha(n-3)[(n-4)(\kappa - f) - 2rf']}{r^2 + \alpha(n-2)[(n-3)(\kappa - f) - rf']} \right], \quad (88)$$

$$K_T(r) := r^{\frac{n}{2}-1} \sqrt{r^2 + \alpha(n-2)[(n-3)(\kappa - f) - rf']}. \quad (89)$$

Here, k_T^2 is an eigenvalue for the LB operator acting on rank two symmetric tensor fields on the horizon manifold σ^n . Using the asymptotic form of $f(r)$, we can calculate the asymptotic behavior of these functions:

$$q_T(r) \sim -\Lambda_{\pm} (2\kappa - k_T^2) \left[\frac{r^2[1 + \alpha\Lambda_{\pm}(n-1)(n-2)] + \alpha\kappa(n-3)(n-4)}{r^2[1 + \alpha\Lambda_{\pm}(n-1)(n-2)] + \alpha\kappa(n-2)(n-3)} \right], \quad r \rightarrow \infty \quad (90)$$

$$K_T(r) \sim r^{\frac{n}{2}-1} \sqrt{r^2[1 + \alpha\Lambda_{\pm}(n-1)(n-2)] + \alpha\kappa(n-2)(n-3)}, \quad r \rightarrow \infty. \quad (91)$$

In principle we have to make a distinction about whether $1 + \alpha\Lambda_{\pm}(n-1)(n-2)$ is zero or not, or equivalently (because of (85)), about whether or not the parameters Λ and α are related in the form $\Lambda = -\frac{n(n+1)}{4\alpha(n-1)(n-2)}$ (which implies that both roots (85) are equal). Note, however, that this distinction is actually irrelevant for the asymptotic behavior of $q_T(r)$, since in both cases this function tends to a constant as $r \rightarrow \infty$. We find:

- (a) If $\Lambda = -\frac{n(n+1)}{4\alpha(n-1)(n-2)}$, then $K_T(r) \sim r^{\frac{n}{2}-1} \sqrt{\alpha\kappa(n-2)(n-3)}$ as $r \rightarrow \infty$, and consequently

$$V_T(r) \sim \text{constant} + \frac{n(n-2)}{4} \Lambda_{\pm}^2 r^2 \sim \text{constant} + \frac{n(n-2)}{4} x^{-2}, \quad r \rightarrow \infty \quad (92)$$

where we have used the asymptotic form (86).

(b) If $\Lambda \neq -\frac{n(n+1)}{4\alpha(n-1)(n-2)}$, then $K_T(r) \sim r^{\frac{n}{2}} \sqrt{1 + \alpha\Lambda_{\pm}(n-1)(n-2)}$ as $r \rightarrow \infty$, and then

$$V_T(r) \sim \text{constant} + \frac{n(n+2)}{4}x^{-2}, \quad r \rightarrow \infty. \quad (93)$$

In both cases the coefficient of the leading term of the potential near $x = 0$ is greater or equal than $\frac{3}{4}$ (recall $n > 2$); then V_T is type 2.4.

2. For vector perturbations, the potential is

$$V_V(r) = q_V(r) + \frac{f}{K_V} \frac{d}{dr} \left(f \frac{dK_V}{dr} \right), \quad (94)$$

where now

$$q_V(r) := \frac{f}{r^2} [k_V^2 - (n-1)\kappa] H(r), \quad (95)$$

$$K_V(r) := \left[r^{\frac{n}{2}-1} \sqrt{r^2 + \alpha(n-2)[(n-3)(\kappa-f) - rf']} \right]^{-1} = \frac{1}{K_T(r)} \quad (96)$$

Here k_V^2 is an eigenvalue for the LB operator on vector field on σ^n and the function $H(r)$ is given by

$$H(r) = \frac{n-3}{2(n-1)} - \frac{n+1}{2(n-1)} \frac{(\Lambda_+ - \Lambda_-)^2}{[2\psi(r) - (\Lambda_+ + \Lambda_-)]^2} \quad (97)$$

We distinguish again two cases:

(a) $\Lambda = -\frac{n(n+1)}{4\alpha(n-1)(n-2)}$: the two roots in (85) are equal, $\Lambda_+ = \Lambda_-$, and $1 + \alpha\Lambda_{\pm}(n-1)(n-2) = 0$. The function $H(r)$ is then a constant, and consequently $q_V(r)$ tends asymptotically to a constant. For $K_V(r)$, we get $K_V(r) \sim [\alpha\kappa(n-2)(n-3)]^{-1/2} r^{-\frac{n}{2}+1}$, $r \rightarrow \infty$, which implies that the asymptotic behavior of the potential is

$$V_V \sim \text{constant} + \frac{(n-2)(n-4)}{4}x^{-2}, \quad r \rightarrow \infty. \quad (98)$$

We are interested in the case $n \geq 3$, since $n = 2$ reduces to the SAdS₄ solution we have already analyzed. We find then that in the classification of Section IV, the potential is type 2.2 for $n = 3$, type 1 for $n = 4$, and type 2.4 for $n \geq 5$.

(b) $\Lambda \neq -\frac{n(n+1)}{4\alpha(n-1)(n-2)}$: we have $1 + \alpha\Lambda_{\pm}(n-1)(n-2) \neq 0$. In this case $H(r)$ also tends asymptotically to a constant, and consequently so does $q_V(r)$. For $K_V(r)$ we find $K_V(r) \sim [1 + \alpha\Lambda_{\pm}(n-1)(n-2)]^{-1/2} r^{-\frac{n}{2}}$, $r \rightarrow \infty$, and the asymptotic form of $V_V(r)$ is

$$V_V \sim \text{constant} + \frac{n(n-2)}{4}x^{-2}, \quad r \rightarrow \infty \quad (99)$$

Since $\frac{n(n-2)}{4} \geq \frac{3}{4}$ for all $n \geq 3$, the potential is type 2.4 in this case.

Table I summarizes the results found in this Section and in Section III.

VI. TYPE 1 POTENTIALS AND ROBIN INSTABILITIES

For type 1 potentials, the local solutions of the differential equation (55) at the conformal boundary are of the form (63). They are well defined at $x = 0$ and so are their derivatives. The possible self adjoint extensions correspond to choosing Dirichlet ($\alpha = \pi/2$ in (63)), Neumann ($\alpha = 0$) or Robin boundary conditions, which are characterized by a parameter $\gamma = \tan(\alpha) \neq 0, \pm\infty$. We denote the corresponding Hamiltonian operators (34) as ${}^D\mathcal{H}$, ${}^N\mathcal{H}$ and ${}^\gamma\mathcal{H}$ respectively. In this Section we study the possibility that a chosen self adjoint extension admits a negative energy

	$d = 4$	$d = 5$	$d = 6$	$d > 6$
EM(V)	1	2.4	2.4	2.4
EM(S)	1	2.2	1	2.4
LGR (T)	-	2.4	2.4	2.4
LGR (V)	1	2.4	2.4	2.4
LGR (S)	1	2.2	1	2.4
LEGB (T)	-	2.4	2.4	2.4
LEGB (V) $\Lambda_+ = \Lambda_-$	-	2.2	1	2.4
LEGB (V) $\Lambda_+ \neq \Lambda_-$	-	2.4	2.4	2.4

TABLE I. Type of the effective potentials for tensor, vector and scalar modes of electromagnetic fields, linearized general relativity and linearized Einstein Gauss-Bonnet gravity around black holes of dimension $d \geq 4$. Note that: i) tensor perturbations exists only for linear gravity in $d \geq 5$; ii) the classification for LEGB vector modes is different when the effective cosmological constants agree (degenerate case) from that in the general, non degenerate case.

eigenfunction ψ_E , $E < 0$, which, according to equation (54) and the comments below it, implies an instability. For such a solution, the asymptotic behavior near the horizon is (61) with $\beta = 0$, then ψ_E is a “bound state”, i.e., ψ_E satisfies

$$-\psi_E'' + V\psi_E = E\psi_E, \quad \text{for } x \in (-\infty, 0), \quad E < 0 \quad (100)$$

and, as follows from (62), belongs to $L^2((-\infty, 0), dx)$:

$$\int_{-\infty}^0 \psi_E^2 dx < \infty. \quad (101)$$

If the chosen self adjoint extension is not Dirichlet, we may normalize ψ_E such that

$$\psi_E(0) = 1, \quad \psi_E'(0) = \gamma. \quad (102)$$

Our intuition from Quantum Mechanics in \mathbb{R} fails for the Schrödinger operator \mathcal{H} on a half line subject to Robin boundary conditions, as this may admit negative eigenvalues even if $V \geq 0$. The reason is that the “kinetic energy” operator $-\partial_x^2$ fails to be positive definite if $\gamma > 0$, as the following simple calculation of the expectation values of \mathcal{H} for a (non necessarily normalized) real, square integrable function on $(-\infty, 0)$ shows:

$$\langle \psi, \mathcal{H}\psi \rangle := \int_{-\infty}^0 \psi \mathcal{H}\psi dx = -\psi\psi'|_{x=0} + \int_{-\infty}^0 [(\psi')^2 + V\psi^2] dx. \quad (103)$$

For Dirichlet or Neumann boundary conditions at $x = 0$ the first term on the right vanishes and we find that $V \geq 0$ does imply $\langle \psi, \mathcal{H}\psi \rangle > 0$, i.e., the self-adjoint extensions ${}^D\mathcal{H}$ and ${}^N\mathcal{H}$ have positive spectra. For a self-adjoint extension ${}^\gamma\mathcal{H}$ corresponding to the Robin boundary condition

$$\psi'(0) = \gamma\psi(0), \quad \gamma \neq 0, \quad (104)$$

we find from (103) that

$$\langle \psi, {}^\gamma\mathcal{H}\psi \rangle = -\gamma \psi(0)^2 + \int_{-\infty}^0 [(\psi')^2 + V\psi^2] dx, \quad (105)$$

which, assuming $V \geq 0$, is positive for $\gamma < 0$, but may be negative if $\gamma > 0$.

A. General results for type 1 potentials

In what follows, we use the fact that any function in the domain of a self adjoint operator such as ${}^\gamma\mathcal{H}$ can be expanded in a basis of generalized eigenfunctions of the operator. Therefore, as in ordinary Quantum Mechanics, there is a function ψ in this domain for which $\langle \psi, {}^\gamma\mathcal{H}\psi \rangle < 0$ if and only if the spectrum of ${}^\gamma\mathcal{H}$ contains a negative value of E .

Proposition 1 (Robin instabilities). *Let V be type 1. For large enough positive γ the spectrum of ${}^\gamma\mathcal{H}$ contains a negative eigenvalue.*

Proof. Since type 1 potentials are bounded, $|V(x)| \leq V_o$ for some $V_o \geq 0$. Consider now the trial function $\psi = e^{\gamma x}$. If $\gamma > 0$, this function belongs to the domain of ${}^\gamma\mathcal{H}$, and the expectation value of ${}^\gamma\mathcal{H}$ is

$$\langle \psi, {}^\gamma\mathcal{H}\psi \rangle = \int_{-\infty}^0 e^{\gamma x} (-\gamma^2 e^{\gamma x} + V e^{\gamma x}) dx = -\frac{\gamma}{2} + \int_{-\infty}^0 V e^{2\gamma x} dx \leq -\frac{\gamma}{2} + \frac{V_o}{2\gamma}, \quad (106)$$

which is negative for

$$\gamma > \sqrt{V_o} \quad (107)$$

□

The following proposition, together with the previous one, tells us that the set of γ such that ${}^\gamma\mathcal{H}$ admits a negative energy eigenvalue, is of the form (γ_c, ∞) (for $\gamma = \gamma_c$ we expect a non square integrable $E = 0$ generalized eigenstate).

Proposition 2. *Let $V(x)$ be type 1. Assume that ${}^{\gamma_o}\mathcal{H}$ admits a negative energy bound eigenstate, then so does ${}^\gamma\mathcal{H}$ for $\gamma > \gamma_o$.*

Proof. Let ψ_o be a negative energy bound eigenstate of ${}^{\gamma_o}\mathcal{H}$, normalized such that $\psi_o(0) = 1$, i.e., ψ_o satisfies (100)-(102) for γ_o . Fix $\alpha \equiv \gamma - \gamma_o > 0$ and let $\phi_\delta(x)$ be a smooth function of $(\delta, x) \in (-\delta_o, \delta_o) \times (-\infty, \delta_o)$ for some positive δ_o such that: $\phi_\delta(-\delta) = 1$, $\phi'_\delta(-\delta) = 0 = \phi''_\delta(-\delta)$, ϕ_δ and ϕ'_δ are growing functions for $x \in (-\delta, 0)$, $\phi'_\delta(0)/\phi_\delta(0) = \alpha$ and $\phi_\delta(0) < 2$. An example of such a function is

$$\phi_\delta(x) = \frac{\alpha(x+\delta)^3}{\delta^2(3-\alpha\delta)} + 1, \quad 0 < \delta < \frac{3}{2\alpha} = \delta_o. \quad (108)$$

Define

$$\psi_\delta(x) = \begin{cases} \psi_o(x) & , x \leq -\delta \\ \psi_o(x)\phi_\delta(x) & , -\delta < x \leq 0 \end{cases} \quad (109)$$

Note that this function belongs to the domain of ${}^\gamma\mathcal{H}$. An example of ψ_o and the corresponding ψ_δ is depicted in Figure 1.

Since V is continuous and bounded, ψ_o and ψ'_o are bounded near $x = 0$, and we may assume that

$$|\psi_o(x)| < A, \quad |\psi'_o(x)| < B, \quad |V(x)| < C \quad \text{for } x \in (-\delta_o, 0). \quad (110)$$

This implies that

$$|\psi_\delta(x)| < 2A, \quad |\psi'_\delta(x)| < 2(B + \alpha A) \quad \text{for } x \in (-\delta_o, 0). \quad (111)$$

From these two equations and $\phi_\delta(0) = (1 - \alpha\delta/3)^{-1}$ follows that

$$\begin{aligned} \langle \psi_\delta, {}^\gamma\mathcal{H}\psi_\delta \rangle - \langle \psi_o, {}^{\gamma_o}\mathcal{H}\psi_o \rangle &= \gamma_o - \gamma\phi_\delta(0)^2 + \int_{-\delta}^0 [(\psi'_\delta)^2 - (\psi'_o)^2 + V(\psi_\delta^2 - \psi_o^2)] \\ &< \gamma_o - \gamma(1 - \alpha\delta/3)^{-2} + \delta [4(B + \alpha A)^2 + B^2 + 5CA^2] \end{aligned} \quad (112)$$

Note that A, B and C are independent of δ , then for δ small enough the right side of (112) is negative. This implies that $\langle \psi_\delta, {}^\gamma\mathcal{H}\psi_\delta \rangle < \langle \psi_o, {}^{\gamma_o}\mathcal{H}\psi_o \rangle < 0$. Since ψ_δ is in the domain of ${}^\gamma\mathcal{H}$, the spectrum of ${}^\gamma\mathcal{H}$ contains a negative energy. □

B. Non negative type 1 potentials

For non negative type 1 potentials a number of useful properties can be easily proved:

Proposition 3. *Let $V(x) \geq 0$ be a type 1 potential. The self-adjoint extensions ${}^D\mathcal{H}$, ${}^N\mathcal{H}$ and ${}^\gamma\mathcal{H}$ with $\gamma < 0$ are positive definite.*

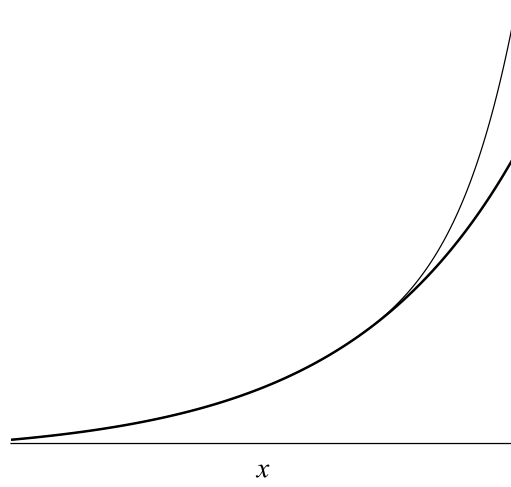


FIG. 1. En example of ψ_o (thick line) and the corresponding ψ_δ (thin line) used in Proposition 2

Proof. This follows from equations (103) and (105). \square

The following proposition shows that if $V(x) \geq 0$ is type 1 and γ positive, ${}^\gamma\mathcal{H}$ admits at most one negative energy eigenstate. It also establishes some properties of the corresponding eigenfunction.

Proposition 4. *Let $V(x) \geq 0$ be a type 1 potential. Assume there is $E < 0$ and ψ_E satisfying (100)-(102), then it follows that:*

- i) ψ_E has no roots, then we can choose it to be positive.
- ii) ψ_E grows monotonically from 0 to $1 = \psi_E(0)$ and ψ'_E grows monotonically from 0 to $\gamma = \psi'_E(0)$ in the interval $x \in (-\infty, 0]$
- iii) There is at most one $E < 0$ and one ψ_E for which conditions (100)-(102) hold.

Proof.

- i) First note that the roots of ψ_E are isolated points: if $x_o = \lim_{n \rightarrow \infty} x_n$ were roots of ψ_E , then $\psi_E(x_o) = 0 = \partial_x \psi_E(x_o)$ and, since ψ_E satisfies the second order equation (100), $\psi_E(x) = 0$ for all x , which is a contradiction. Note also that there cannot exist a sequence x_n of consecutive roots such that $\lim_{n \rightarrow \infty} x_n = -\infty$, otherwise, there would be a sequence x'_n of positive local maxima of ψ_E , $\lim_{n \rightarrow \infty} x'_n = -\infty$, and/or a sequence x''_n of negative local minima of ψ_E , $\lim_{n \rightarrow \infty} x''_n = -\infty$. Both cases lead to a contradiction: take, e.g., a sequence x'_n of positive maxima: the conditions $\psi_E(x'_n) > 0$, $\partial_x^2 \psi_E(x'_n) < 0$, $\lim_{n \rightarrow \infty} x'_n = -\infty$ give

$$0 \geq \lim_{n \rightarrow \infty} \frac{\partial_x^2 \psi_E(x'_n)}{\psi_E(x'_n)} = \lim_{j \rightarrow \infty} (V(x'_n) - E) = -E, \quad (113)$$

which contradicts $E < 0$ (if we assume a sequence x''_n of negative minima we get the same contradiction as, again, $\partial_x^2 \psi_E(x''_n)/\psi_E(x''_n) < 0$). We may therefore assume, without loss of generality (i.e., replacing ψ_E with $-\psi_E$ if necessary), that $\psi_E > 0$ for large negative x . Now assume ψ_E has roots, then there is a root x_o with largest absolute value (the least upperbound of the non empty bounded set $\{z \leq 0 | \psi(x) > 0 \text{ for } x \in (-\infty, z)\}$). Since ψ_E is square integrable, $\lim_{x \rightarrow -\infty} \psi_E(x) = 0 = \psi_E(x_o)$, and $\psi_E(x) > 0$ in the interval $x \in (-\infty, x_o)$. It follows that there is a local maximum at $x_1 < x_o$, and this leads us back to the same type of contradiction as above: $0 > \partial_x^2 \psi_E(x_1)/\psi_E(x_1) = V(x_1) - E$, however $V(x_1) - E > 0$.

- ii) As ψ_E is in the domain of ${}^\gamma\mathcal{H}$, $\int_{-\infty}^0 (\psi'_E)^2 dx < \infty$, then $\lim_{x \rightarrow -\infty} \psi'_E(x) = 0$. From the hypotheses and i) follows that $\psi_E > 0$ for $x \in (-\infty, 0]$, then $\partial_x^2 \psi_E = (V - E)\psi_E > 0$. This implies that $\partial_x \psi_E$ grows monotonically from zero to $\partial_x \psi_E(0)$. As $\partial_x \psi_E > 0$ everywhere, ψ_E increases monotonically from zero to $\psi_E(0)$.

- iii) Assume $0 > E_2 > E_1$ are two eigenvalues of ${}^\gamma\mathcal{H}$. In view of ii) we may assume that $\psi_{E_i}(x) > 0$ for all x , $i = 1, 2$. Equation (100) implies that the Wronskian

$$W = (\partial_x \psi_{E_1})\psi_{E_2} - (\partial_x \psi_{E_2})\psi_{E_1} \quad (114)$$

satisfies $\partial_x W = (E_2 - E_1)\psi_{E_2}\psi_{E_1} > 0$ for all x . This implies that W grows monotonically. However $\lim_{x \rightarrow -\infty} W(x) = 0 = W(0)$. The uniqueness of ψ_E follows from (102). \square

Proposition 5. *Let $V(x) \geq 0$ be a type 1 potential and consider the operator ${}^\gamma\mathcal{H}$ for $\gamma > 0$.*

- i) *If V is non trivial, for small enough γ the spectrum of ${}^\gamma\mathcal{H}$ contains no negative eigenvalue.*
- ii) *There is a critical value $\gamma_c > 0$ such that the set $\{\gamma \mid \text{the spectrum of } {}^\gamma\mathcal{H} \text{ contains a negative eigenvalue}\}$ is of the form (γ_c, ∞) .*
- iii) *For $\gamma > \gamma_c$, the negative energy eigenvalue E_γ satisfies $|E_\gamma| \leq 2\gamma^2$.*
- iv) *If $\int_{-\infty}^0 V dx < \infty$ then $\gamma_c \leq 2 \int_{-\infty}^0 V dx$.*

Proof.

- i) Assume ${}^\gamma\mathcal{H}$ admits a negative energy E_γ and let ψ_{E_γ} be the eigenstate satisfying $\psi_{E_\gamma}(0) = 1$. From Lemma 4 we know that ψ_{E_γ} is positive, convex and monotonically growing in $(-\infty, 0]$. In particular, $\psi_{E_\gamma}(x) > \gamma x + 1$ for $-1/\gamma < x < 0$ (Figure 2), then

$$\gamma = \int_{-\infty}^0 \psi_{E_\gamma}''(x) dx = \int_{-\infty}^0 (V(x) - E_\gamma)\psi_{E_\gamma}(x) dx > \int_{-1/\gamma}^0 (V(x) - E_\gamma)(\gamma x + 1) dx = h(\gamma) + k(\gamma), \quad (115)$$

where, for $\gamma > 0$ we defined

$$h(\gamma) = \int_{-1/\gamma}^0 V(x)(\gamma x + 1) dx \geq 0 \quad \text{and} \quad k(\gamma) = -\frac{E_\gamma}{2\gamma} > 0. \quad (116)$$

Note that $dh/d\gamma = \int_{-1/\gamma}^0 V(x)x dx \leq 0$, then h is a positive, decreasing function of γ , diverging as $\gamma \rightarrow 0^+$ unless $\int_{-\infty}^0 V(x) dx$ is finite. It follows from (115) and (116) that the existence of a negative energy eigenvalue implies $\gamma > h(\gamma)$, and this implies that $\gamma > \gamma_*$, where $\gamma_* > 0$ is the only solution of $\gamma = h(\gamma)$. We conclude that, for small positive γ , ${}^\gamma\mathcal{H}$ is positive definite.

- ii) This follows from i) and Propositions 1 and 2. For $\gamma = \gamma_c$ we expect the lowest energy eigenvalue to be $E = 0$.
- iii) From (115) and (116) follows that, for the bound state, $\gamma \geq k(\gamma) = -E_\gamma/(2\gamma)$.
- iv) For the trial function used in Proposition 1

$$\langle \psi, {}^\gamma\mathcal{H}\psi \rangle = -\frac{\gamma}{2} + \int_{-\infty}^0 V e^{2\gamma x} dx < -\frac{\gamma}{2} + \int_{-\infty}^0 V dx, \quad (117)$$

and this is negative for

$$\gamma > 2 \int_{-\infty}^0 V dx. \quad (118)$$

\square

For potentials with finite integral, condition (118) assures the existence of negative energy states for ${}^\gamma\mathcal{H}$. The sharper bound

$$\gamma > \int_{-\infty}^0 V dx. \quad (119)$$

was proved in [8] using different methods. We can prove (119) by making the assumption that the spectrum of ${}^\gamma\mathcal{H}$ is continuous in γ

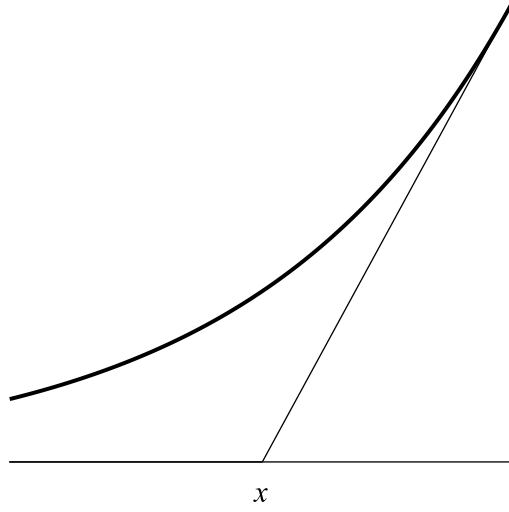


FIG. 2. Graphs of ψ_{E_γ} (thick line) and $\gamma x + 1$ for $x > -1/\gamma$, zero otherwise (thin line) used in Proposition 5.i).

Proposition 6. *Let $V(x) \geq 0$ be a type 1 potential and assume that $\lim_{\gamma \rightarrow \gamma_c^+} E_\gamma = 0$, then*

$$\gamma_c \leq \int_{-\infty}^0 V(x) dx, \quad (120)$$

Proof. For $\gamma > \gamma_c$ and the normalization $\psi_{E_\gamma}(0) = 1$

$$\gamma = \int_{-\infty}^0 (V(x) - E_\gamma) \psi_{E_\gamma}(x) dx. \quad (121)$$

Taking the limit $\gamma \rightarrow \gamma_c^+$ and using Proposition 4.ii) this gives

$$\gamma_c = \lim_{\gamma \rightarrow \gamma_c^+} \int_{-\infty}^0 (V(x) - E_\gamma) \psi_{E_\gamma}(x) dx = \lim_{\gamma \rightarrow \gamma_c^+} \int_{-\infty}^0 V(x) \psi_{E_\gamma}(x) dx \leq \int_{-\infty}^0 V(x) dx, \quad (122)$$

from where the existence of a negative energy in the spectrum for γ satisfying (119) follows. \square

C. Energy considerations

As pointed out in [3], no matter which self adjoint extension we choose for \mathcal{H} , there is always a notion of conserved energy for solutions of the wave equation (52)-(53) in the domain $x < 0$ of 1+1 Minkowski spacetime for the solution (54)-(55). If ${}^z\mathcal{H}$ is the chosen self adjoint extension ($z = D, N, \gamma$), the energy is defined as

$$\mathcal{E}_z = \frac{1}{2} (\langle \partial_t \phi, \partial_t \phi \rangle + \langle \phi, {}^z\mathcal{H} \phi \rangle). \quad (123)$$

Conservation of \mathcal{E}_z follows from $[\partial_t, {}^z\mathcal{H}] = 0$ and the self adjointness of ${}^z\mathcal{H}$:

$$\dot{\mathcal{E}}_z = \langle \partial_t \phi, \partial_t^2 \phi \rangle + \frac{1}{2} (\langle \partial_t \phi, {}^z\mathcal{H} \phi \rangle + \langle \phi, \partial_t {}^z\mathcal{H} \phi \rangle) = \langle \partial_t \phi, \partial_t^2 \phi + {}^z\mathcal{H} \phi \rangle = 0. \quad (124)$$

Integrating (123) by parts we get (c.f. equation (103))

$$\mathcal{E}_z = -\frac{1}{2} z \phi^2 \Big|_{x=0} + \frac{1}{2} \int_{-\infty}^0 [(\partial_t \phi)^2 + (\partial_x \phi)^2 + V \phi^2] dx \equiv \mathcal{E}_o - \frac{1}{2} z \phi^2 \Big|_{x=0}, \quad (125)$$

where $z = 0$ for Dirichlet or Neumann boundary conditions and $z = \gamma = (\partial_x \phi / \phi)|_{x=0}$ for Robin boundary conditions.

The conservation of \mathcal{E}_z in (125) can also be derived by applying Gauss' theorem to the conserved current $J_a = T_{ab}(\partial/\partial t)^b$, where T_{ab} is the energy momentum tensor of the 1+1 field theory (52)-(53) in the two dimensional (half) Minkowski space. The region of integration is the one bounded by two $t = \text{constant}$ surfaces. The integral at the horizon ($x = -\infty$) vanishes for fields ϕ which are square integrable on t -slices, so if we use inertial coordinates (t, x) we get

$$0 = \int_{-\infty}^0 T_{tt}(t, x) dx - \int_{-\infty}^0 T_{tt}(t_o, x) dx - \int_{t_o}^t T_{tx}(t', 0) dt', \quad (126)$$

where $T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} \eta_{ab} (\partial_c \phi \partial^c \phi + V \phi^2)$. Note that

$$T_{tt} = \frac{1}{2} (\dot{\phi}^2 + \phi'^2 + V \phi^2), \quad (127)$$

and $T_{tx} = \dot{\phi} \phi'$, then

$$\lim_{x \rightarrow 0^-} T_{tx} = \lim_{x \rightarrow 0^-} \dot{\phi} \phi' \rightarrow \begin{cases} 0 & , \text{N or D boundary conditions} \\ \frac{1}{2} \gamma \partial_t (\phi^2) & , \text{Robin boundary conditions.} \end{cases} \quad (128)$$

Thus, for N or D boundary condition the third term in (126) vanishes, there is no flux of energy at the conformal boundary, and the canonical energy \mathcal{E}_o defined in (125) is conserved. As seen from equation (128), the form (104) of the Robin boundary condition is crucial for the existence of the conserved quantity $z = \gamma$ in (125), as it allows the flux of energy at infinity (the timelike boundary) to be integrated, reducing (126) to

$$0 = \mathcal{E}_o(t) - \mathcal{E}_o(t_o) - \frac{\gamma}{2} (\phi(t, 0))^2 + \frac{\gamma}{2} (\phi(t_o, 0))^2, \quad (129)$$

from where the conservation of \mathcal{E}_z for the $z = \gamma$ case of (125) follows. Note from (128) and (129) that the change $\gamma \rightarrow -\gamma$ reverses the sign of the energy flow at the timelike boundary.

D. Toy models

1. Vanishing potential: a semi infinite string

Consider a string extending along the $x < 0$ half axis and oscillating in the (x, y) plane, and let $\phi(x, t)$ be the y -displacement at time t . For small $\partial_x \phi$ we use the standard approximation $\partial_x \phi = \tan(\alpha) \simeq \sin(\alpha)$ for the angle α of the string with respect to the horizontal at x and calculate the vertical component of the tension T as $T_y \simeq T \partial_x \phi$, then Newton's law applied to the piece of string extending from x to $x + \Delta x$ gives

$$T [\partial_x \phi(x + \Delta x, t) - \partial_x \phi(x, t)] = \rho \Delta x \partial_t^2 \phi, \quad (130)$$

where ρ is the mass per unit length. Taking the limit $\Delta x \rightarrow 0$ we obtain the wave equation

$$\frac{1}{c^2} \partial_t^2 \phi - \partial_x^2 \phi = 0, \quad c^2 = T/\rho, \quad (131)$$

which, after rescaling t , has the form (52)-(53) with $\mathcal{H} = -\partial_x^2$, that is $V = 0$.

For a string with a right end at $x = 0$, Newton's law applied to the $-\Delta x < x < 0$ piece of the string

$$f - T \partial_x \phi(0 - \Delta x, t) = \rho \Delta x \partial_t^2 \phi, \quad (132)$$

gives, after taking the limit $\Delta x \rightarrow 0$, the force f applied to the string end at $x = 0$:

$$f = T \partial_x \phi(0, t). \quad (133)$$

Dirichlet boundary conditions $\phi(0, t) = 0$ corresponds to fixing the string at the $x = 0$ end, Neumann boundary conditions $\partial_x \phi(0, t) = 0$ to leaving it free ($f = 0$), and Robin boundary conditions $\partial_x \phi(0, t) = \gamma \phi(0, t)$ to subjecting the end of the string to an elastic force

$$f = \gamma T \phi(0, t) \quad (134)$$

which corresponds to a spring with elastic constant $k = -\gamma T$ if $\gamma < 0$, and gives a “repulsive elastic force” if $\gamma > 0$.

Dirichlet boundary conditions: The eigenfunctions of ${}^D\mathcal{H}$ are

$$\psi_{(E=k^2)} = \sqrt{\frac{2}{\pi}} \sin(kx), \quad k > 0. \quad (135)$$

These were normalized such that

$$\langle \psi_{(E=k^2)}, \psi_{(E=l^2)} \rangle = \frac{2}{\pi} \int_{-\infty}^0 \sin(kx) \sin(lx) dx = \delta(k-l), \quad (k, l > 0). \quad (136)$$

Functions ψ satisfying the Dirichlet boundary conditions extend naturally to odd functions in \mathbb{R} . The completeness of the above set of eigenfunctions then follows from the sine Fourier transform theorem for odd functions:

$$\psi(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty A_k \sin(kx) dk, \quad (137)$$

where

$$A_k = \sqrt{\frac{2}{\pi}} \int_{-\infty}^0 \psi(x) \sin(kx) dx \quad (138)$$

The conserved energy in this case is \mathcal{E}_o defined in (125). Energy can be transferred among the different normal modes of the string but remains conserved. This is due to the fact that the external force that keeps the $x = 0$ end of the string fixed does no work on the string.

Neumann boundary conditions: The eigenfunctions of ${}^N\mathcal{H}$ are

$$\psi_{(E=k^2)} = \sqrt{\frac{2}{\pi}} \cos(kx), \quad k > 0. \quad (139)$$

These were normalized such that

$$\langle \psi_{(E=k^2)}, \psi_{(E=l^2)} \rangle = \frac{2}{\pi} \int_{-\infty}^0 \cos(kx) \cos(lx) dx = \delta(k-l), \quad (k, l > 0). \quad (140)$$

Functions ψ satisfying the Neumann boundary conditions are naturally extended to even functions in \mathbb{R} . The completeness of the above set of eigenfunctions then follows from the cosine Fourier transform theorem of even functions. The conserved energy in this case is again \mathcal{E}_o defined in (125). Vibrational energy can be transferred among the different normal modes of the string but remains conserved. This is due to the fact that no external force is acting on the $x = 0$ end of the string.

Robin boundary conditions: For any value of γ , the generalized eigenfunctions corresponding to the continuum spectrum are

$$\psi_{(E=k^2)} = \sqrt{\frac{2}{\pi}} \left(1 + \frac{k^2}{\gamma^2}\right)^{-1/2} \left[\sin(kx) + \frac{k}{\gamma} \cos(kx) \right], \quad k > 0. \quad (141)$$

These were normalized such that

$$\int_{-\infty}^0 \psi_{(E=k^2)} \psi_{(E=l^2)} dx = \delta(k-l). \quad (142)$$

To verify (142) we use the integrals (136) and (140) together with the distributional identity

$$\int_{-\infty}^0 \sin(lx) \cos(kx) dx = \frac{l}{(k+l)(k-l)}. \quad (143)$$

From Proposition 5.iv we expect instabilities when $\gamma > 0$. This is trivially verified, the only (Proposition 4.iii) bound state of $\gamma\mathcal{H}$ for $\gamma > 0$ is

$$\psi_{(E=-\gamma^2)} = \sqrt{2\gamma} e^{\gamma x}, \quad (144)$$

where we have normalized such that the integral of $\psi_{(E=-\gamma^2)}^2$ equals one. This function has to be added to the set (141) to form a complete orthonormal set for the domain of ${}^\gamma\mathcal{H}$ when $\gamma > 0$. To verify that (144) is orthogonal to the functions (141) we use

$$\int_{-\infty}^0 e^{\gamma x} e^{-ikx} dx = \frac{1}{\gamma - ik}, \quad \gamma > 0. \quad (145)$$

We can also understand the expansion of functions satisfying (104) in terms of ordinary Fourier transforms by means of the following observation: the function $\tau = \psi - \partial_x \psi / \gamma$ vanishes at $x = 0$, and can naturally be extended to an odd function in \mathbb{R} , for which the sine Fourier representation is possible:

$$\tau(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tau_k \sin(kx) dk, \quad \tau_k = \sqrt{\frac{2}{\pi}} \int_{-\infty}^0 \tau(x) \sin(kx) dx. \quad (146)$$

ψ can be easily recovered from τ :

$$\psi(x) = e^{\gamma x} \left[\psi(0) + \gamma \int_x^0 \tau(u) e^{-\gamma u} du \right] \quad (147)$$

If we use the representation (146) of τ in the above equation we arrive at the following expansion of ψ in terms of $\psi_{(E=-\gamma^2)}$ and the $\psi_{(E=k^2)}$ above:

$$\psi(x) = \left[\psi(0) - \sqrt{\frac{2}{\pi}} \int_0^\infty dk \frac{\gamma k \tau_k}{\gamma^2 + k^2} \right] e^{\gamma x} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\gamma^2 \tau_k}{\gamma^2 + k^2} \left(\sin(kx) + \frac{k}{\gamma} \cos(kx) \right) dk. \quad (148)$$

It is a non trivial fact that the coefficient between square brackets above vanishes if $\gamma < 0$.

The conserved energy \mathcal{E}_γ (equation (125)) contains two terms: the string vibrational energy \mathcal{E}_o and the potential energy $-\gamma\phi^2(x=0, t)/2$ of the force acting on the $x=0$ end of the string. For negative γ , this is an ordinary elastic force pulling towards $\phi=0$: we may imagine that the $x=0$ end of the string is attached to a spring of elastic constant $-\gamma$, then energy flows from the string to the spring and viceversa in such a way that the total energy \mathcal{E}_γ remains constant. Since the spring potential energy is positive definite, the amount of energy the spring can transfer to the string is finite; this keeps the string vibrations bounded. For positive γ , instead, the force at the $x=0$ end is “repulsive elastic”, pushing away the string end with an intensity that *increases* as $\phi(x=0)$ grows. The repulsive elastic potential $-\gamma\phi^2/2$ is unbounded from below, it can feed the string with an unlimited amount of energy and produce unbounded oscillations.

The string analogy can be extended to the $V(x) \geq 0$ case by assuming that, besides the string tension, there is an x -dependent restoring elastic force pulling the string to the $\phi=0$ configuration. In this case, instead of (130) we have

$$T [\partial_x \phi(x + \Delta x, t) - \partial_x \phi(x, t)] - V(x) T \Delta x \phi(x, t) = \rho \Delta x \partial_t^2 \phi \quad (149)$$

which, taking the limit $\Delta x \rightarrow 0$ in (149) and rescaling t gives (52)-(53).

2. A step potential

Consider now the case [?]]

$$V(x) = \begin{cases} 0 & x < -a \\ V_o & a < x \leq 0 \end{cases} \quad (150)$$

where V_o and a are positive, and assume ${}^\gamma\mathcal{H}$ has a negative energy eigenvalue $E = -\alpha^2, \alpha > 0$. The wave function is proportional to

$$\psi(x) = \begin{cases} \exp(\alpha(x+a)) & , x < -a \\ \cosh(\beta(x+a)) + \frac{\alpha}{\beta} \sinh(\beta(x+a)) & , -a < x \leq 0 \end{cases} \quad (151)$$

where

$$\beta = \sqrt{\alpha^2 + V_o}. \quad (152)$$

This function is C^1 and satisfies the properties in Proposition 4. Imposing (104) on (151) gives the following relation

$$\gamma = \beta \frac{\beta \tanh(\beta a) + \alpha}{\beta + \alpha \tanh(\beta a)} \quad (153)$$

Inserting (152) in (153) we find that $d\gamma/d\alpha > 0$ for positive α . Since $\gamma \sim \alpha$ for $\alpha \rightarrow \infty$, we conclude that, as α goes from zero to infinity, γ ranges from

$$\gamma_c = \sqrt{V_o} \tanh(\sqrt{V_o} a) \quad (154)$$

to infinity. This example is useful because γ_c introduced in Proposition 5.ii can be explicitly calculated, equation (154). If $\gamma > \gamma_c$, (153) has a unique solution α , which gives a unique bound state, of energy $E = -\alpha^2$. On the other hand, if $\gamma \leq \gamma_c$, there are no bound states. The bounds (107) and (119) can be easily checked in this example; moreover, the example shows that (119) cannot be improved as, for small a (154) gives

$$\gamma_c = aV_o + \mathcal{O}(a^3) = \int_{-\infty}^0 V dx + \mathcal{O}(a^3). \quad (155)$$

VII. ROBIN INSTABILITIES IN SADS₄

In this section we explore briefly the instabilities of Maxwell fields and linearized gravity on SAdS₄. For the Maxwell field, starting from (41)-(42), we show that the energy \mathcal{E}_o in (125) (multiplied times $\ell(\ell+1)$ and summed over ℓ) agrees with the field energy obtained by integrating the energy density of the Maxwell field on a t -slice, and the $\gamma\phi^2(t, x=0)/2$ term in (125) comes from the ℓ piece contribution of the energy flux from infinity. We expect a similar situation in the gravity case, where $T_{\alpha\beta}$ is the effective energy-momentum tensor, quadratic in the first order fields, which appears source the second order perturbation equations, however, this calculation becomes unwieldy even using symbolic manipulation computing. For even gravitational perturbations we show the explicitly known [4] unstable solution for a particular γ .

A. Maxwell fields

The energy momentum tensor of the Maxwell field

$$T_{\alpha\beta} = \frac{1}{4\pi}(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}) \quad (156)$$

gives a conserved current $J_{\alpha} := T_{\alpha\beta}\xi^{\beta}$ for any Killing field ξ^{α} . Taking $\xi^{\beta}\partial/\partial x^{\beta} = \partial/\partial t$ and applying Gauss' theorem to J^{α} in a region Ω of the spacetime limited by two $t = \text{constant}$ surfaces gives

$$0 = \int_{\Omega} \nabla_{\alpha} J^{\alpha} = \int_{\partial\Omega} J_{\alpha} n^{\alpha} = \int_{\Sigma_t} J_{\alpha} n_{\Sigma_t}^{\alpha} + \int_{\Sigma_{t_o}} J_{\alpha} n_{\Sigma_{t_o}}^{\alpha} + \int_{\mathcal{I}_t} J_{\alpha} n_{\mathcal{I}}^{\alpha} \quad (157)$$

where \mathcal{I}_t is the $R \rightarrow \infty$ limit of an $r = R$ hypersurface extending from Σ_{t_o} to Σ_t , $t > t_o$. The induced volume elements on the hypersurfaces are understood on the integrals, the outer pointing unit normal vectors are $n_{\Sigma_t}^{\alpha} \partial/\partial x^{\alpha} = f^{-1/2} \partial/\partial t$, $n_{\Sigma_t}^{\alpha} \partial/\partial x^{\alpha} = -f^{-1/2} \partial/\partial t$ and $n_{\mathcal{I}_t}^{\alpha} \partial/\partial x^{\alpha} = f^{1/2} \partial/\partial r$. We can therefore rewrite (157) as

$$\begin{aligned} \int_{-\infty}^0 dx \int_{S^2} T_{tt}(t, r, \theta, \phi) \sin \theta d\theta d\phi &= \int_{-\infty}^0 dx \int_{S^2} T_{tt}(t_o, r, \theta, \phi) \sin \theta d\theta d\phi \\ &+ \lim_{R \rightarrow \infty} R^2 f(R) \int_{t_o}^t dt \int_{S^2} T_{tr}(t, R, \theta, \phi) \sin(\theta) d\theta d\phi, \end{aligned} \quad (158)$$

the second line giving the failure for the “standard energy” to be conserved. Inserting $A_{\beta} = A_{\beta}^{(-)} + A_{\beta}^{(+)}$ given in (41)-(42) into $F = dA$ and (156) gives (a prime denotes $\partial_x = f\partial_r$)

$$(J_{\alpha}^{(\ell, m)} n_{\mathcal{I}}^{\alpha})|_{r=R} = \sqrt{f} T_{tr}|_{r=R} = \frac{1}{4\pi R^2 \sqrt{f(R)}} \left[\dot{\phi}_{(\ell, m)}^{+} \phi_{(\ell, m)}^{\prime +} + \dot{\phi}_{(\ell, m)}^{-} \phi_{(\ell, m)}^{\prime -} \right] \left[(\partial_{\theta} S_{(\ell, m)})^2 + \frac{1}{\sin^2 \theta} (\partial_{\phi} S_{(\ell, m)})^2 \right]. \quad (159)$$

Note that in the $R \rightarrow \infty$ limit $\dot{\phi}_{(p,\ell,m)}\phi'_{(p,\ell,m)} = \gamma_{(p,\ell,m)}\phi_{(p,\ell,m)}^2$ ($\gamma_{(p,\ell,m)}$ the Robin constant (104) for the mode ($p = \pm, \ell, m$)). This fact allows to integrate the flux at \mathcal{I}_t . Note also that changing the sign of $\gamma_{(p,\ell,m)}$ reverses the direction of the flux at \mathcal{I}_t .

To proceed we use that

$$\begin{aligned} \int_{S^2} \left[(\partial_\theta S_{(\ell,m)})^2 + \frac{1}{\sin^2 \theta} (\partial_\phi S_{(\ell,m)})^2 \right] \sin(\theta) d\theta d\phi \\ = \int_{S^2} \hat{D}^A S_{(\ell,m)} \hat{D}_A S_{(\ell,m)} = - \int_{S^2} S_{(\ell,m)} \hat{D}^A \hat{D}_A S_{(\ell,m)} = 4\pi\ell(\ell+1), \end{aligned} \quad (160)$$

where \hat{D}_A is the covariant derivative on S^2 and we used the orthonormality of the $S_{(\ell,m)}$. After a lengthy calculation, we find that (158) reduces to

$$\begin{aligned} \sum_{(\ell,m,p=\pm)} \frac{\ell(\ell+1)}{2} \int_{-\infty}^0 \left[\dot{\phi}_{(p,\ell,m)}^2 + \phi_{(p,\ell,m)}'^2 + V_\ell^{Max} \phi_{(p,\ell,m)}^2 \right] dx \Big|_t = \\ \sum_{(\ell,m,p=\pm)} \frac{\ell(\ell+1)}{2} \int_{-\infty}^0 \left[\dot{\phi}_{(p,\ell,m)}^2 + \phi_{(p,\ell,m)}'^2 + V_\ell^{Max} \phi_{(p,\ell,m)}^2 \right] dx \Big|_{t_o} \\ + \sum_{(\ell,m,p=\pm)} \frac{\ell(\ell+1)}{2} \gamma_{(p,\ell,m)} (\phi_{(p,\ell,m)}(s,0))^2 \Big|_{s=t_o}^{s=t}. \end{aligned} \quad (161)$$

The conservation (161) also follows from (125), which gives, for every (p, ℓ, m)

$$\frac{1}{2} \int_{-\infty}^0 \left[\dot{\phi}_{(p,\ell,m)}^2 + \phi_{(p,\ell,m)}'^2 + V_\ell^{Max} \phi_{(p,\ell,m)}^2 \right] dx = \text{constant} + \frac{1}{2} \gamma_{(p,\ell,m)} (\phi_{(p,\ell,m)}(t,0))^2. \quad (162)$$

The left hand side above will remain bounded for negative $\gamma_{(p,\ell,m)}$. For positive $\gamma_{(p,\ell,m)}$ instabilities are allowed where $(\phi_{(p,\ell,m)}(t,0))^2$ increases without bound together with the canonical energy \mathcal{E}_o .

Equation (47) shows that the Maxwell potentials V_ℓ^{Max} have a finite integral over x so, according to (119) the associated field will be unstable for $\gamma_{(p,\ell,m)} > \int_{-\infty}^0 V_\ell^{Max} dx = \frac{\ell(\ell+1)}{r_h}$.

B. Linearized gravity

1. Explicit unstable modes

The following field was reported in [4] as an unstable solution for the even gravitational perturbations of SAdS₄ (equations (52)-(53) with potential (49)) satisfying Robin boundary conditions:

$$\phi_{(\ell,m)}^{+ \text{ unst}} = \chi_\ell^+(r) \exp(w_\ell t) \quad (163)$$

where

$$\chi_\ell^+(r) = \frac{r \exp(w_\ell x)}{(\ell+2)(\ell-1)r + 6M}, \quad (164)$$

x is the radial coordinate defined in (4), and

$$w_\ell = \frac{1}{12M} \frac{(\ell+2)!}{(\ell-2)!}. \quad (165)$$

$\chi_\ell^+(r)$ defined in (164) satisfies $\mathcal{H}^+ \chi_\ell^+ = -w_\ell^2 \chi_\ell^+$ where \mathcal{H}^+ is the Hamiltonian for even/scalar gravitational perturbations. This equation is satisfied for any value of M and Λ , as long as $x(r)$ in (164) satisfies $dx/dr = 1/f$ (c.f. equation (4)) with the appropriate parameters. For $\Lambda = 0$, this solution was found by Chandrasekhar [9] when looking for

algebraically special perturbations: those with the property that the first order variation of one of the Weyl scalars Ψ_0 or Ψ_4 vanishes. A linearly independent solution with the same negative energy is [9]

$${}_{r_o}\tau_\ell^+(r) = \chi_\ell^+(r) \int_{r_o}^r \frac{dr'}{f(r')(\chi_\ell^+(r'))^2}. \quad (166)$$

Note that changing r_o above adds a term proportional to $\chi_\ell^+(r)$.

Two linearly independent solutions of $\mathcal{H}^-\psi_\ell^- = -w_\ell^2\psi_\ell^-$ for the odd (vector) Hamiltonian with this same negative energy $E = -w_\ell^2$ are [9]

$$\chi_\ell^-(r) = \frac{1}{\chi_\ell^+(r)}, \quad {}_{r_o}\tau_\ell^-(r) = \chi_\ell^-(r) \int_{r_o}^r \frac{dr'}{f(r')(\chi_\ell^-(r'))^2}. \quad (167)$$

The fact that, for $\Lambda \geq 0$, x ranges from minus infinity as $r \rightarrow r_h^+$, to infinity as $r \rightarrow \infty$ ($\Lambda = 0$) or approaches the cosmological horizon ($\Lambda > 0$), makes the algebraically special perturbations (164), (166) and (167) uninteresting in these cases because all these solutions diverge at at least one of these two limits, and so are irrelevant as they are not eigenfunctions of \mathcal{H}^\pm . The situation is different for $\Lambda < 0$ and also for the non globally hyperbolic Schwarzschild naked singularity ($\Lambda = 0, M < 0$). For the latter, x can be chosen to range from $x = 0$ (the timelike boundary at the $r = 0$ singularity) to infinity (as $r \rightarrow \infty$) and, as $w_\ell < 0$ in this case, (164) behaves properly in both limits (this happens only for even perturbation, neither $\chi^-(r)$ nor $\tau_{r_o}^-(r)$ for any r_o behaves properly). Moreover, it was found in [12] (see also [11]) that there is a single boundary condition at the $r = 0$ timelike boundary that leads to a consistent linear perturbation treatment; therefore, the dynamics is not ambiguous in spite of the non globally hyperbolic character of the spacetime. This particular Robin boundary condition is precisely the one satisfied by the mode (164). Since this mode grows exponentially in time, the claim that the Schwarzschild naked singularity is unstable is free of ambiguities [11], [13].

In what follows we concentrate on the $\Lambda < 0$ case, for which $x \in (-\infty, 0)$, and one can check using (5) that $\chi^+(r)$ and

$$\tau_\ell^- \equiv {}_{r_h}\tau_\ell^-(r) = \chi_\ell^-(r) \int_{r_h}^r \frac{dr'}{f(r')(\chi_\ell^-(r'))^2} \quad (168)$$

satisfy the bound state, negative energy requirements (100) and (101) for $\mathcal{H}_{(\ell,m)}^+$ and $\mathcal{H}_{(\ell,m)}^-$ respectively, the choice $r_o = r_h$ above being crucial for this to hold.

The unstable even solution (163) satisfies a Robin boundary condition (104) at $x = 0$ with γ equal to

$$\gamma_{Ch} \equiv w_\ell - \frac{2M\Lambda}{(\ell-1)(\ell+2)} = \frac{\Lambda r_h(\Lambda r_h^2 - 3)}{3(\ell-1)(\ell+2)} - \frac{(\ell+2)(\ell+1)\ell(\ell-1)}{2r_h(\Lambda r_h^2 - 3)} \quad (169)$$

The perturbation of the metric is

$$h_{(\ell,m)}^{+unst} = \exp(w_\ell v) S_{(\ell,m)} \left[\frac{w_\ell}{6M} (r\ell(\ell+1) - 6M) dv \otimes dv + \frac{\ell(\ell+1)}{6M} r^2 (d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi) \right] \quad (170)$$

where we defined $v = t + x$. The above expression shows that the perturbation is well behaved across the future event horizon, defined by $r = r_h$, $v \in \mathbb{R}$. This perturbation has the property of splitting only one of the two pairs of principal null directions of the background so that the perturbed spacetime is Petrov type-II. For generic perturbations, instead, both pairs of principal null directions are split leaving a type-I spacetime (see [14] for details). Besides the principal null direction splitting, the effect of even perturbations on the background geometry is measured by the gauge invariant combination of perturbed curvature scalars [4]

$$G_+ = (9M - 4r + \Lambda r^3)\delta Q_+ + 3r^3\delta X, \quad (171)$$

where

$$X = \frac{1}{720} (\nabla_\epsilon C_{\alpha\beta\gamma\delta}) (\nabla^\epsilon C^{\alpha\beta\gamma\delta}), \quad Q_+ = \frac{1}{48} C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}, \quad (172)$$

$C_{\alpha\beta\gamma\delta}$ is the Weyl tensor and δ in (171) denotes first order variation.

It was shown in [4] that G_+ contains all the gauge invariant information of generic even perturbations and that, when the linearized Einstein's equation are satisfied,

$$G_+ = -\frac{2M \delta M}{r^5} + \frac{M}{2r^4} \sum_{\ell \geq 2} \frac{(\ell+2)!}{(\ell-2)!} [f \partial_r + Z_\ell] \phi_{(\ell,m)}^+ S_{(\ell,m)}, \quad (173)$$

where δM is the mass variation that comes from the $\ell = 0$ even perturbation, and

$$Z_\ell = \frac{2M \Lambda r^3 + \mu r(r-3M) - 6M^2}{r^2(\mu r + 6M)}, \quad \mu = (\ell-1)(\ell+2). \quad (174)$$

For the pure mode (163), (173) gives

$$G_+[\phi_{(\ell,m)}^{+ \text{ unst}}] = \frac{(\ell+2)!}{(\ell-2)!} \left(\frac{\ell(\ell+1)r - 6M}{24r^5} \right) \exp(w_\ell v), \quad v = t + x(r). \quad (175)$$

The existence of the bound state χ_ℓ^+ for the Zerilli SAdS₄ Hamiltonian (52)-(53), which has a positive, type 1 potential (49), implies, in view of Proposition 5.iii, that for even ℓ modes the critical value of γ for instabilities satisfies

$$\gamma_c \leq \gamma_{Ch} \simeq \begin{cases} \frac{(\ell+2)!}{(\ell-2)!} \frac{1}{6r_h} & , r_h \rightarrow 0 \\ \frac{\Lambda^2 r_h^3}{3(\ell-1)(\ell+2)} & , r_h \rightarrow \infty, \end{cases} \quad (176)$$

We can use this to test the upper bound (122) in these limits using (51):

$$\gamma_c < \int_{-\infty}^0 V_\ell^{(+)} dx \simeq \begin{cases} \frac{2\ell^2+2\ell-3}{2r_h} & , r_h \rightarrow 0 \\ \frac{2\Lambda^2 r_h^3}{3(\ell+2)(\ell-1)} & , r_h \rightarrow \infty. \end{cases} \quad (177)$$

For large horizon radius γ_{Ch} is half the value of the integrated potential, whereas for small r_h we find that γ_{Ch} is less than the integrated potential only for $\ell = 2$ (the minimum possible ℓ value), and grows as ℓ^4 for large ℓ , whereas the integrated potential grows only as ℓ^2 . We should keep in mind, however that the statement (122) cannot be improved, as the $a \rightarrow 0$ limit of the step potential example in Section VID 2 saturates this inequality.

2. Boundary conditions and even/odd duality breaking

Unlike the negative mass Schwarzschild solution, for which a single boundary condition at the conformal timelike boundary is singled out from the infinite set of z -conditions ($z = D, N$ or γ) by a consistency requirement of the linear perturbation scheme [12]; for SAdS₄, Dirichlet, Neumann or Robin boundary conditions with γ below the critical value *for every* (ℓ, m) would give a stable dynamics, whereas a single odd mode for which γ is high enough would introduce an instability (we recall that the potential for even perturbations is type 1 and positive definite, then Propositions 1 to 6 apply to them, whereas odd perturbations have a type 1 potential that is negative near the horizon for small enough r_h/M , then only Propositions 1-3 apply to them in general.) This implies that any gravitational stability claim for SAdS₄ is meaningless unless an specification of the chosen boundary conditions is made.

The suitability of a boundary condition depends on the problem at hand, and in the linear gravity case it should be kept in mind that the fields $\phi_{(\ell,m)}^\pm$, although convenient to disentangle the linearized Einstein equations, are not directly measurable quantities. For odd perturbations, the field $G_- = \delta Q_-$, where

$$Q_- = \frac{1}{48} {}^* C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}, \quad (178)$$

${}^* C^{\alpha\beta}{}_{\gamma\delta}$ the Hodge dual of the Weyl tensor, is a gauge invariant (and thus measurable) scalar that accounts for the distortion of the geometry caused by an odd perturbation and from where the odd part of the metric perturbation in a given gauge can be recovered [4]. On shell, the relation between δQ_- and the $\phi_{(\ell,m)}^-$ is [4]

$$G_- = -\frac{6M}{r^7} \sqrt{\frac{4\pi}{3}} \sum_{m=1}^3 j^{(m)} S_{(\ell=1,m)} - \frac{3M}{r^5} \sum_{\ell > 1, m} \frac{(\ell+2)!}{(\ell-2)!} \phi_{(\ell,m)}^-, \quad (179)$$

where $j^{(m)}$, $m = 1, 2, 3$ are the components of the perturbed black hole angular momentum. Similarly, the curvature related field G_+ defined in (173) satisfies the same requirements of being gauge invariant and encoding all the information of the *even* piece of a generic metric perturbation. Thus, perturbations can be analyzed entirely using the curvature related gauge invariants G_\pm in (173) and (179). Imposing Dirichlet conditions on these fields is equivalent to requiring mixed Dirichlet/Robin conditions on the $\phi_{(\ell,m)}^\pm$:

$$\phi_{(\ell,m)}^- \Big|_{x=0} = 0, \quad \frac{\partial_x \phi_{(\ell,m)}^+}{\phi_{(\ell,m)}^+} \Big|_{x=0} = -\frac{2M\Lambda}{(\ell-1)(\ell+2)}, \quad (180)$$

whereas imposing Neumann or Robin boundary conditions on the G_\pm gives Robin conditions on the $\phi_{(\ell,m)}^\pm$.

Another context where Robin boundary conditions on the $\phi_{(\ell,m)}^\pm$ arise is that of the AdS/CFT correspondence, under which the unperturbed background corresponds to a perfect fluid in the boundary and one is interested in metric perturbations that, in a preferred gauge, vanish in large r =constant surfaces [22] [25].

In what follows, we proceed to analyze the relation between boundary conditions at the conformal boundary $r = \infty$ and the formal duality exchanging odd and even modes, discovered by Chandrasekhar about thirty years ago [9] [10]. All the relations we need follow from the observations in [9] [10] that

$$\mathcal{H}_\ell^\pm = \mathcal{D}_\ell^\pm \mathcal{D}_\ell^\mp - w_\ell^2, \quad (181)$$

where

$$\mathcal{D}_\ell^\pm = \pm \partial_x + W_\ell, \quad W_\ell = w_\ell + \frac{6Mf}{r(r\mu + 6M)}, \quad (182)$$

and also that

$$W_\ell = \frac{\partial_x \chi_\ell^+}{\chi_\ell^+} = -\frac{\partial_x \chi_\ell^-}{\chi_\ell^-}, \quad (183)$$

which can be verified using (164) and (167). From these we find that

$$V_\ell^+ = W_\ell' + (W_\ell)^2 - w_\ell^2 = \frac{\chi_\ell^{+''}}{\chi_\ell^+} - w_\ell^2, \quad (184)$$

$$V_\ell^- = -W_\ell' + (W_\ell)^2 - w_\ell^2 = \frac{\chi_\ell^{-''}}{\chi_\ell^-} - w_\ell^2 \quad (185)$$

where a prime means derivative with respect to x . The second form in (184) allows us to write the ordinary differential equation $\mathcal{H}_\ell^+ \psi = E\psi$ as

$$-\frac{\psi''}{\psi} + \frac{\chi_\ell^{+''}}{\chi_\ell^+} = (E + w_\ell^2) \quad (186)$$

from where, for $E = -w_\ell^2$, we readily obtain the two linearly independent solutions $\psi = \chi_\ell^+$, and $\psi = \tau_\ell^+$ given in (166). A similar analysis leads to the unstable odd modes (167).

In view of (181), acting with \mathcal{D}_ℓ^- on a solution to the differential equation $\mathcal{H}_\ell^+ \psi^+ = E\psi^+$ gives a -possibly trivial- solution $\psi^- = \mathcal{D}_\ell^- \psi^+$ of $\mathcal{H}_\ell^- \psi^- = E\psi^-$ and viceversa:

$$\mathcal{H}_\ell^+ \psi^+ = E\psi^+ \Rightarrow \mathcal{H}_\ell^- (\mathcal{D}_\ell^- \psi^+) = E(\mathcal{D}_\ell^- \psi^+), \quad (187)$$

$$\mathcal{H}_\ell^- \psi^- = E\psi^- \Rightarrow \mathcal{H}_\ell^+ (\mathcal{D}_\ell^+ \psi^-) = E(\mathcal{D}_\ell^+ \psi^-). \quad (188)$$

From the equations above we find some trivial cases:

$$\mathcal{D}_\ell^- \chi_\ell^+ = 0, \quad \mathcal{D}_\ell^+ \chi_\ell^- = 0, \quad (189)$$

(note that the most general solution of the equation $\mathcal{D}_\ell^- \psi^+ = 0$ ($\mathcal{D}_\ell^+ \psi^- = 0$) is a constant times χ_ℓ^+ (χ_ℓ^-)); the fact that these functions satisfy $\mathcal{H}_\ell^\pm \chi_\ell^\pm = -w_\ell^2 \chi_\ell^\pm$ is a consequence that otherwise the effect of \mathcal{D}_ℓ^\pm can be reversed by \mathcal{D}_ℓ^\mp (times a function of E), and viceversa:

$$\mathcal{H}_\ell^+ \psi^+ = E \psi^+ \Rightarrow \mathcal{D}_\ell^+ (\mathcal{D}_\ell^- \psi^+) = (\mathcal{H}_\ell^+ + w_\ell^2) \psi^+ = (E + w_\ell^2) \psi^+ \neq 0 \quad (190)$$

$$\mathcal{H}_\ell^- \psi^- = E \psi^- \Rightarrow \mathcal{D}_\ell^- (\mathcal{D}_\ell^+ \psi^-) = (\mathcal{H}_\ell^- + w_\ell^2) \psi^- = (E + w_\ell^2) \psi^- \neq 0. \quad (191)$$

From equations (190) (191) follows that if $\psi_j^-, j = 1, 2$, are two linearly independent solutions of $\mathcal{H}_\ell^- \psi_j^- = E \psi_j^-$ with $E \neq -w_\ell^2$, then $\psi_j^+ = \mathcal{D}_\ell^+ \psi_j^-, j = 1, 2$, are two linearly independent solutions of $\mathcal{H}_\ell^+ \psi_j^+ = E \psi_j^+$, and similarly if we exchange $-$ and $+$.

We also note from the above equations that

$$\mathcal{D}_\ell^- r_o \tau_\ell^+ = \chi_\ell^-, \quad \mathcal{D}_\ell^+ r_o \tau_\ell^- = \chi_\ell^+ \quad (192)$$

for any r_o , and that

$$\mathcal{D}_\ell^+ \kappa_\ell^- = \chi_\ell^+ \Rightarrow \kappa_\ell^- = \tau_\ell^- + \alpha \chi_\ell^-, \quad (193)$$

where τ_ℓ^- was defined in (168) and α is a constant.

The possibility of exchanging even and odd modes using \mathcal{D}_ℓ^\pm is the duality, peculiar to four dimensions, that we will analyze for $\Lambda < 0$ in the remaining part of this Section. For $\Lambda \geq 0$, the fields $\phi_{(\ell, m)}^\pm$ belong to $L^2(\mathbb{R}, dx)$ and the operators \mathcal{D}_ℓ^\pm give a bijection between the sets of solutions of the odd and even 1+1 wave equations (see Section 4.5 in [4].) The case where $\Lambda < 0$ is much subtler. The linear gravity potentials V_ℓ^\pm are type 1, the values at $x = 0$ of solutions of $\mathcal{H}_\ell^\pm \psi^\pm = E \psi^\pm$ and their x -derivatives are well defined and generically non-zero (equation (63)), so Dirichlet, Neumann or Robin boundary conditions are allowed. Suppose that, for a given (ℓ, m) , we choose

$$\psi^{+'}|_{x=0} = \gamma_e \psi^+|_{x=0} \quad (194)$$

where, in what follows $\gamma_e \in \mathbb{R} \cup \{\infty\}$ to include the cases $\gamma_e = 0$ (Neumann) and $\gamma_e = \infty$ (Dirichlet), and we similarly introduce γ_o for the odd modes, dropping the (ℓ, m) indices for simplicity.

From (182) we find that

$$(\mathcal{D}_\ell^- \psi^+)|_{x=0} = -\psi^{+'}|_{x=0} + (W_\ell \psi^+)|_{x=0} = (W_\ell - \gamma_e) \psi^+|_{x=0} \quad (195)$$

and that

$$(\mathcal{D}_\ell^- \psi^+)'|_{x=0} = -\psi^{+''}|_{x=0} + (W'_\ell + \gamma_e W_\ell) \psi^+|_{x=0}, \quad (196)$$

where, from (183)

$$W|_{x=0} = \gamma_{Ch} \quad \text{and} \quad W'|_{x=0} = \left(\frac{2M\Lambda}{\mu} \right)^2, \quad (197)$$

and γ_{Ch} was defined in (169). It is easy to prove from these two equations that, in general, there is no function $\gamma_o(\gamma_e)$, $\gamma_e, \gamma_o \in \mathbb{R} \cup \{\infty\}$, such that $\psi^{+'}/\psi^+ = \gamma_e$ at $x = 0$ implies $(\mathcal{D}_\ell^- \psi^+)'/\mathcal{D}_\ell^- \psi^+ = \gamma_o(\gamma_e)$ at $x = 0$. To show this, we use the fact that ψ^+ fields satisfying (194) can be expanded using the complete basis of generalized eigenfunctions ${}^{\gamma_e} \psi_E^+$ (we suppress the ℓ index) of the corresponding self adjoint extension ${}^{\gamma_e} \mathcal{H}_\ell^+$, so that

$$\psi^+ = \int dE c_E {}^{\gamma_e} \psi_E^+, \quad (198)$$

where the integral notation includes a sum over bound states, if there were any.

For an energy eigenstate we find from (196) that

$$(\mathcal{D}_\ell^- {}^{\gamma_e} \psi_E^+)'|_{x=0} = (\mathcal{H}_\ell^+ - V_\ell^+ + W'_\ell + \gamma_e W_\ell) {}^{\gamma_e} \psi_E^+|_{x=0} \quad (199)$$

$$= (E + w_\ell^2 - W_\ell^2 + \gamma_e W_\ell) {}^{\gamma_e} \psi_E^+|_{x=0}, \quad (200)$$

which, together with (194) gives

$$\left. \frac{(\mathcal{D}_\ell^- \gamma_e \psi_E^+)' }{\mathcal{D}_\ell^- \gamma_e \psi_E^+} \right|_{x=0} = \left. \frac{E + w_\ell^2 - W_\ell^2 + \gamma_e W_\ell}{W_\ell - \gamma_e} \right|_{x=0}. \quad (201)$$

Since, generically, the quotient above depends on E , $\psi^{+'}/\psi^+$ in (198) will have different values for different functions in the linear space obtained by applying \mathcal{D}_ℓ^- to the domain of $\gamma_e \mathcal{H}_\ell^+$, then the dynamics will not be defined in this space since it is not a self adjoint domain of \mathcal{H}_ℓ^- [?]. The only exceptions (i.e., situations where the right hand side of (201) does not depend on E) are: i) when we choose Dirichlet boundary conditions in the even sector, that is $\gamma_e \rightarrow \infty$ in (201), which gives Robin conditions in the odd sector with $\gamma_o = -W|_{x=0} = -\gamma_{Ch}$, that is

$$\gamma_o = -\gamma_{Ch} \quad (\gamma_e = \infty), \quad (202)$$

and ii) when we choose Dirichlet boundary conditions in the odd sector, that is, $\gamma_e = W|_{x=0} = \gamma_{Ch}$ in (201)):

$$\gamma_e = \gamma_{Ch} \quad (\gamma_o = \infty). \quad (203)$$

The following proposition gives more details about the supersymmetry and these two cases:

Proposition 7. *Consider the maps \mathcal{D}^\pm defined in (182). In what follows we use the symbol $\gamma \mathcal{H}_\ell^\pm$ both for the self adjoint operator and its domain.*

- i) *The spectra of ${}^D\mathcal{H}_\ell^+$ and ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ are nonnegative.
 $\mathcal{D}_\ell^- : {}^D\mathcal{H}_\ell^+ \rightarrow {}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ is a bijection.*
- ii) *The spectrum of ${}^D\mathcal{H}_\ell^-$ is nonnegative, that of ${}^{\gamma_{Ch}}\mathcal{H}_\ell^+$ contains a negative energy with eigenfunction χ^+ .
The map $\mathcal{D}_\ell^- : {}^{\gamma_{Ch}}\mathcal{H}_\ell^+ \rightarrow {}^D\mathcal{H}_\ell^-$ is surjective and has kernel the linear space generated by χ_ℓ^+ .
The map $\mathcal{D}_\ell^+ : {}^D\mathcal{H}_\ell^- \rightarrow {}^{\gamma_{Ch}}\mathcal{H}_\ell^+$ is injective.*
- iii) *There are no other values of $\gamma, \gamma' \in \mathbb{R} \cup \{\infty\}$ such that $\mathcal{D}_\ell^\mp(\gamma \mathcal{H}_\ell^\pm) \subset \gamma' \mathcal{H}_\ell^\mp$*

Proof. We have already proven iii).

To prove i) note that the only solution of $\mathcal{D}_\ell^- \psi^+ = 0$ is a constant times χ^+ , which does not belong to ${}^D\mathcal{H}_\ell^+$, so the map in i) has a trivial kernel and therefore is injective. Similarly, the only solution of $\mathcal{D}_\ell^+ \psi^- = 0$ is a constant times $\chi^- = 1/\chi^+$ which, although satisfies a Robin condition with $\gamma = -\gamma_{Ch}$ at $x = 0$, diverges as $x \rightarrow -\infty$ and so does not belong to ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$. This implies that the map $\mathcal{D}_\ell^+ : {}^{-\gamma_{Ch}}\mathcal{H}_\ell^- \rightarrow {}^D\mathcal{H}_\ell^+$ is injective. Since V_ℓ^+ is nonnegative, Proposition 3 applies and the spectrum of ${}^D\mathcal{H}_\ell^+$ is nonnegative. The spectrum of ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ must also be nonnegative, otherwise, a negative eigenfunction of ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ would be sent by the injective map \mathcal{D}_ℓ^+ to a negative eigenfunction of ${}^D\mathcal{H}_\ell^+$, which is a contradiction. This proves that both ${}^D\mathcal{H}_\ell^+$ and ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ have nonnegative spectra and therefore lead to stable dynamics in the even and odd sectors respectively. To prove that $\mathcal{D}_\ell^- : {}^D\mathcal{H}_\ell^+ \rightarrow {}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ is onto we proceed as in Lemma 7 in [4]: let ψ^- be an arbitrary function in ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ and $\psi^- = \int dE q(E) {}^{-\gamma_{Ch}}\psi_E^-$ its expansion in eigenfunctions ${}^{-\gamma_{Ch}}\psi_E^-$ of ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$. Since the spectrum of ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$ is nonnegative,

$$\tilde{\psi}^- = \int dE \left(\frac{q(E)}{w_\ell^2 + E} \right) {}^{-\gamma_{Ch}}\psi_E^- \quad (204)$$

is well defined and belongs to ${}^{-\gamma_{Ch}}\mathcal{H}_\ell^-$. Then $\mathcal{D}_\ell^+ \tilde{\psi}^-$ is in ${}^D\mathcal{H}_\ell^+$ and is the function sent to ψ^- by \mathcal{D}_ℓ^- , as the following calculation shows:

$$\mathcal{D}_\ell^- (\mathcal{D}_\ell^+ \tilde{\psi}^-) = (\mathcal{H}_\ell^- + w_\ell^2) \int dE \left(\frac{q(E)}{w_\ell^2 + E} \right) {}^{-\gamma_{Ch}}\psi_E^- = \psi^- \quad (205)$$

This completes the proof of i).

To prove ii) recall that the only solution of $\mathcal{D}_\ell^+ \psi^- = 0$ is a constant times χ^- , which does not belong to ${}^D\mathcal{H}_\ell^-$, therefore $\mathcal{D}_\ell^+ : {}^D\mathcal{H}_\ell^- \rightarrow {}^{\gamma_{Ch}}\mathcal{H}_\ell^+$ is injective. We have already proven that the kernel of $\mathcal{D}_\ell^- : {}^{\gamma_{Ch}}\mathcal{H}_\ell^+ \rightarrow {}^D\mathcal{H}_\ell^-$ is the one dimensional space of functions proportional to χ_ℓ^+ , which is the eigenfunction of the only (Proposition 4.iii) negative

energy in the spectrum of ${}^{\gamma_{Ch}}\mathcal{H}_\ell^+$. If ${}^D\mathcal{H}_\ell^-$ admitted a negative energy, an eigenfunction κ_ℓ^- of this energy would be sent to a negative energy eigenfunction of ${}^{\gamma_{Ch}}\mathcal{H}_\ell^+$ by the injective map \mathcal{D}_ℓ^+ ; i.e., we may assume that $\mathcal{D}_\ell^+\kappa_\ell^- = \chi_\ell^+$. However, it follows (193) that the general solution of the differential equation $\mathcal{D}_\ell^+\kappa = \chi_\ell^+$ is $\tau_\ell^- + \alpha\chi_\ell^-$ and, given that $\tau_\ell^-(x=0) > 0$, we need $\alpha \neq 0$ for $\tau_\ell^- + \alpha\chi_\ell^-$ to equal zero at $x=0$ and, since $\alpha \neq 0$, the resulting function diverges as $x \rightarrow -\infty$ and therefore does not belong to ${}^D\mathcal{H}_\ell^-$, so we reach a contradiction and conclude that ${}^D\mathcal{H}_\ell^-$ has a nonnegative spectrum. This allows us to prove that $\mathcal{D}_\ell^- : {}^{\gamma_{Ch}}\mathcal{H}_\ell^+ \rightarrow {}^D\mathcal{H}_\ell^-$ is surjective proceeding as above: expand ${}^D\mathcal{H}_\ell^- \ni \psi^- = \int dE s(E) {}^D\psi_E^-$ in a basis of generalized eigenfunctions ${}^D\psi_E^-$ of the positive definite operator ${}^D\mathcal{H}_\ell^-$. Consider the function

$$\hat{\psi}^- = \int dE \left(\frac{s(E)}{w_\ell^2 + E} \right) {}^D\psi_E^-. \quad (206)$$

Note that $\mathcal{D}_\ell^+\hat{\psi}^-$ belongs to the domain of ${}^{\gamma_{Ch}}\mathcal{H}_\ell^+$ and that \mathcal{D}_ℓ^- sends it to ψ^- . \square

VIII. SUMMARY

In what follows we enumerate the subjects addressed and the results proven in this work on asymptotically anti de Sitter black holes:

1. Stability of scalar field as an indicator of gravitational stability

The stability of a scalar test field on a given spacetime is oftentimes taken as indicative of linear gravitational stability. SAdS₄ offers an example of how naive this idea can be: although there is a single choice of boundary condition for a scalar field on SAdS₄, under which the field is stable, there are infinitely many possible dynamics for gravitational perturbations, and they give quite different results. If any mixture of Dirichlet, Neumann or Robin boundary conditions with γ below the critical value is chosen for the different modes, the evolution of gravitational perturbations will be stable. If, on the contrary, a Robin boundary condition with γ above the critical value is allowed for a single (ℓ, m) , the perturbation will be unstable, the instability being signaled by an exponentially growing mode similar to (170). The same statements hold for Maxwell fields.

2. Form of the effective potentials for generalized gravity in arbitrary dimensions

For Einstein or Lovelock black holes in higher dimensions with constant curvature horizon manifolds, the modal decomposition reduces the problem of scalar, Maxwell and linear gravity perturbations equations to the form (52)-(53) which is wave equation in the $x < 0$ half of 1+1 Minkowski spacetime. We have found that in all cases the potentials $V(x)$ in equation (53) fit into the type 1 / type 2 classification introduced in Section IV, equations (59) and (60), as shown in Table I. In dimension four the potentials are type 1 (not diverging at $x=0$), this implies that the functional space of solutions of (52)-(53) are functions ϕ for which both ϕ and $\partial_x\phi$ are defined at the conformal boundary at $x=0$ and on which we can impose, Dirichlet ($\phi=0$), Neumann ($\partial_x\phi=0$) or Robin ($\partial_x\phi=\gamma\phi$) conditions at the boundary.

3. Naturalness of Robin boundary conditions

Robin boundary conditions on the 1+1 auxiliary fields (52)-(53) are a natural choice, as they result from the imposition of Dirichlet or Neumann conditions on more relevant fields, as discussed in Section VII B 2. This condition also arises in the context of AdS-CFT dualities, and if we want to preserve the even/odd supersymmetric like duality in SAdS₄ (see below).

4. Robin instabilities

For type 1 potentials there are instabilities for high enough Robin parameter γ . If the potential is nonnegative, there is a critical value $\gamma_c > 0$ such that the set of unstable boundary conditions is $\gamma > \gamma_c$. We prove a number of properties about the energy spectrum and the form of the unstable modes. The mechanism triggering instabilities is explained with the simple toy models of Section VI D. Although for any positive γ energy is pumped into the system from the conformal boundary, we show that this is not enough to trigger instabilities, as γ_c is strictly positive in type 1 positive

potentials, and so there are stable cases with $0 < \gamma < \gamma_c$. SAdS₄ is the simplest black hole solution exhibiting Robin instabilities.

5. Even/odd duality in 4 dimensions

Four dimensional Schwarzschild black holes exhibit a unique feature of a duality exchanging even and odd modes, which is due to the fact that the corresponding potentials form a supersymmetric pair. This was used to extend to the even sector the proof of nonmodal stability for Schwarzschild black holes when $\Lambda \geq 0$. In the asymptotically AdS case, however, the even/odd duality is compromised by the need of mapping boundary conditions preserved by the dynamics. We have found that there are only two boundary conditions compatible with the even/odd symmetry (Dirichlet in one sector and Robin with a particular γ in the other one), and only one of them leads to a stable dynamics.

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